Lesson 1: Working with matrix algebra **Suggested Teaching Time:** 22 hours

Learning Outcome: 1 Be able to use matrix algebra to solve engineering problems.

Topic	Suggested Teaching	Suggested Resources

AC1.1 Perform **operations** in matrix algebra
AC 1.2 Evaluate the determinants of a matrix
AC 1.3 Solve simultaneous equations using **matrix methods**
AC 1.4 Obtain the inverse of a square matrix
AC 1.5 Apply matrix algebra to solve engineering problems described by sets of simultaneous equations
	Delivery should include whole-class teaching to confirm the learners have with a thorough grounding in the matrix arithmetic processes, followed by individual practice in solving typical matrix problems.
	Session 1 (4 hours): Revision of matrix operations
	Before commencing the unit proper, learners should undertake a comprehensive revision of basic matrix algebra. A number of points should be noted:
	• Matrix notation: learners should use large round brackets instead of square ones, to avoid any confusion with the straight vertical lines denoting the matrix determinant
	• Elements of some of the matrices to be worked with should be complex numbers
	• Learners should spend time consolidating the techniques of:
	  • Summation of matrices e.g.:
	$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} (a+p) & (b+q) \\ (c+r) & (d+s) \end{pmatrix}$$
	  • Subtraction of matrices e.g.:
	$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} (a-p) & (b-q) \\ (c-r) & (d-s) \end{pmatrix}$$
	  • **Multiplication of matrices** by a **scalar constant** e.g.:
	$$if\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ then\ 2A = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix}$$
		Book
		Bird. J. O., *Higher Engineering Mathematics* 7th edition (Routledge 2014)
		ISBN-13: 978-0415662826
		Websites:
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City & Guilds

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	So for example: if $A = \begin{pmatrix} 6 & 3 \\ -1 & 2 \end{pmatrix}, B = \begin{pmatrix} 10 & 2 \\ 3 & -7 \end{pmatrix}$ $and\ C = \begin{pmatrix} 4 & -2 \\ -4 & 2 \end{pmatrix}$ $find\ 2A - 3B + 5C$ $$2A = \begin{pmatrix} 12 & 6 \\ -2 & 4 \end{pmatrix}, 3B = \begin{pmatrix} 30 & 6 \\ 9 & -21 \end{pmatrix}\ and\ 5C = \begin{pmatrix} 20 & -10 \\ -20 & 10 \end{pmatrix}$$ $$2A - 3B + 5C = \begin{pmatrix} 12 & 6 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} 30 & 6 \\ 9 & -21 \end{pmatrix} + \begin{pmatrix} 20 & -10 \\ -20 & 10 \end{pmatrix}$$ $$= \begin{pmatrix} 2 & -10 \\ -31 & 35 \end{pmatrix}$$ • Finding the **product of two or more matrices** up to 3 by 3. So for a 2x2 matrix: $$if\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ and\ A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$ then $$A \times B = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$ Tutors should stress that, except in special cases, **A x B ≠ B x A** and learners should be able to demonstrate the standard proof.	

Lesson 1: Working with matrix algebra **Suggested Teaching Time:** 22 hours

Learning Outcome: 1 Be able to use matrix algebra to solve engineering problems.

Topic	Suggested Teaching	Suggested Resources
	Session 2 (4 hours): Evaluating matrix determinants Working with 2 by 2 and 3 by 3 matrices, learners should be proficient in defining and calculating the: • **Determinant** of 2 by 2 matrix using $\mathbf{A \times A^{-1} = I}$ and by i. Interchanging the elements on the leading diagonal (\) ii. Changing the sign of the other two elements iii. Multiply the new matrix by the reciprocal of the determinant of the original matrix • **Determinant** of a 3 by 3 matrix by: i. Finding the minor of each element by covering the rows and column containing the element and obtaining a 2 by 2 determinant ii. Finding the **signed minor** or **cofactor** of the of the element using the sign pattern: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ iii. Finding the **value of the 3 by 3 determinant** which is: **the sum of the products of the elements and their cofactors (signed minors) of any row or any column of the corresponding 3 by 3 matrix**. Thus there are always six ways of evaluating the determinant of a 3 by 3 matrix; the answers will all be the same.	

Lesson 1: Working with matrix algebra **Suggested Teaching Time:** 22 hours

Learning Outcome: 1 Be able to use matrix algebra to solve engineering problems.

Topic	Suggested Teaching	Suggested Resources
	• The **adjoint** of a 3 by 3 matrix - definition and how to obtain it • The **reciprocal** of a 3 by 3 matrix using $\mathbf{A}^{-1} = \frac{\mathbf{adj\ a}}{\mathbf{\|A\|}}$ • **Transposition of matrices** • Defining and using the **identity or unit matrices I_2, I_3, I_4** etc. **Session 3 (5 hours): Matrix methods** Having spent sufficient time practising the techniques of manipulating matrices, this powerful tool will be ready for use. Learners should now progress to the techniques of solving simultaneous equations using matrix methods. In particular the following should be covered: • Solution by determinants • Cramer's rule • Gaussian elimination systematic elimination by row transformations in the augmented matrix • Eigenvalues and eigenvectors.	

Lesson 1: Working with matrix algebra **Suggested Teaching Time:** 22 hours

Learning Outcome: 1 Be able to use matrix algebra to solve engineering problems.

Topic	Suggested Teaching	Suggested Resources
	Session 4 (3 hours): Inverses of square matrices Finally for this learning outcome, learners should focus on methods of obtaining the inverse of a matrix by: • Formula • Row transformations **Session 5 (6 hours): Contextual problems** Also at this point, problems in an engineering context appropriate to the learner's specialisation should be introduced covering subjects such as: • Structural analysis (forces, vectors, mass, tension, loads etc.) • Electrical circuit analysis • Vibration analysis (coupled oscillations) Considerable time should be spent by the learner in exploring the types of engineering problems which can solved using matrices	

Lesson 2: Using vectors **Suggested Teaching Time:** 10 hours

Learning Outcome: 2. Be able to use vector methods to solve engineering problems

Topic	Suggested Teaching	Suggested Resources

AC 2.1 Perform **operations** with vectors
 	
AC 2.2 Solve engineering problems using vectors
	Delivery is likely to be via whole-class teaching initially however the majority of learner activity will be in group and self-study.
	
	Session 1 (3 hours): Revision of vector and scalar basics
	Tutors should ensure firstly that the learners have a good grounding in vector notation and vector resolution. There should be a period of revision of the following in an engineering context:
	• Scalar and vector definition
	• Drawing vectors, including addition by both 'nose to tail' and parallelogram methods
	• Finding resultants of two or more forces by resolution into horizontal and vertical components
	• Addition of vectors by calculation
	• Subtraction of vectors
	• Velocity problems
	• i, j, k notation
	
	Session 2 (3 hours): Vector and scalar products
	
	The learner should then progress to vector and scalar products, with related engineering problems being introduced as soon as is appropriate. Subjects and techniques which should be covered are:
	• The unit vector and unit triad, definitions and calculations:
	  ○ The unit vector for oa is $\frac{\boldsymbol{oa}}{\lvert\boldsymbol{oa}\rvert}$ where oa is the vector and $\lvert oa\rvert$ is the magnitude of the vector
	  ○ The unit triad consists of three unit vectors at right angles to each other
		Book:
		
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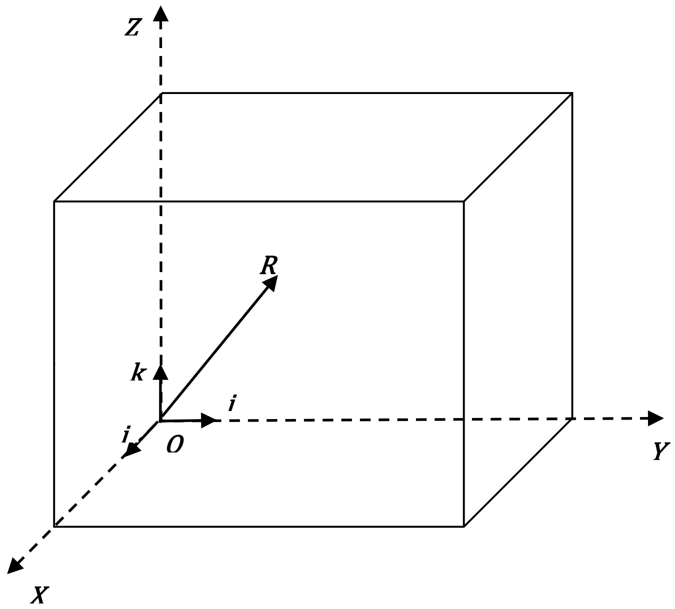
		ISBN-13: 978-0415662826
		
		Website
		
		http://mathworld.wolfram.com/
		
		http://www.mathcentre.ac.uk/

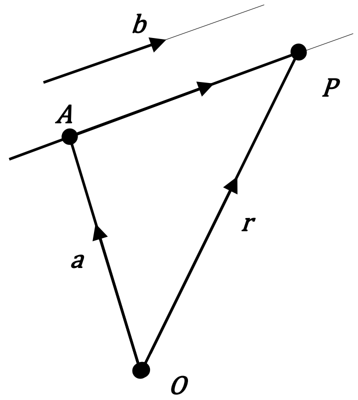
Lesson 2: Using vectors **Suggested Teaching Time:** 10 hours

Learning Outcome: 2. Be able to use vector methods to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	• Derive the expression for the 3-dimensional movement of an object in space from an origin (O) to a point (r) in space in terms of ***i***, ***j***, and ***k***  ***OR = xi +yj = zk***	

Lesson 2: Using vectors **Suggested Teaching Time:** 10 hours

Learning Outcome: 2. Be able to use vector methods to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	• The scalar or dot product of two vectors e.g.: $\boldsymbol{oa} \cdot \boldsymbol{ob} = \boldsymbol{oa} \cdot \boldsymbol{ob} \cdot \cos(\theta_2 - \theta_1)$ where $\theta_2 > \theta_1$ • Vector or cross product: $\lvert \boldsymbol{oa} \cdot \boldsymbol{ob} \rvert = \boldsymbol{oa} \cdot \boldsymbol{ob} \cdot \sin\theta$ • The vector equation of a line, passing through points A and P, in Cartesian form:  By vector addition $\boldsymbol{OP} = \boldsymbol{OA} + \boldsymbol{AP}$ i.e. $\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{AP}$ **(1)** However since the straight line through $\boldsymbol{A}$ is parallel to the free vector $\boldsymbol{b}$ (free vector: one which has the same magnitude, direction and sense), then $\boldsymbol{AP} = \lambda\boldsymbol{b}$ where λ is a scalar quantity. So, from the working above, $\boldsymbol{r} - \boldsymbol{a} = \lambda\boldsymbol{b}$	

Lesson 2: Using vectors **Suggested Teaching Time:** 10 hours

Learning Outcome: 2. Be able to use vector methods to solve engineering problems

<table>
<tr><th>Topic</th><th>Suggested Teaching</th><th>Suggested Resources</th></tr>
<tr><td></td><td>
Assume that $r = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, a = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $b = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ then from equation **(1):**

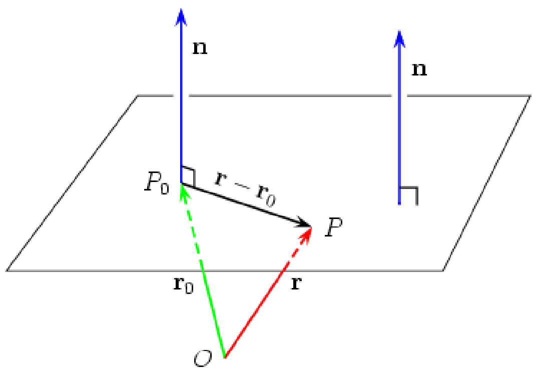
$$x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) + \boldsymbol{\lambda}\,(b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k})$$

$$\frac{\boldsymbol{x - a_1}}{\boldsymbol{b_1}} = \frac{\boldsymbol{y - a_2}}{\boldsymbol{b_2}} = \frac{\boldsymbol{z - a_3}}{\boldsymbol{b3}} = \boldsymbol{\lambda}$$

So $x = a_1 + \lambda b_1,\ y = a_2 + \lambda b_2$ and $z = a_3 + \lambda b_3$
Solving for λ gives:

$$\frac{\boldsymbol{x - a_1}}{\boldsymbol{b_1}} = \frac{\boldsymbol{y - a_2}}{\boldsymbol{b_2}} = \frac{\boldsymbol{z - a_3}}{\boldsymbol{b3}} = \boldsymbol{\lambda}$$

- Vector equation of a plane:



A vector equation of this plane is: $(\boldsymbol{r} - \boldsymbol{r_0}) \cdot \boldsymbol{n} = \boldsymbol{0}$
</td><td>http://www.maths.usyd.edu.au/u/MOW/vectors/vectors-13/v-13-1.html</td></tr>
</table>

Lesson 2: Using vectors **Suggested Teaching Time:** 10 hours

Learning Outcome: 2. Be able to use vector methods to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Session 3 (4 hours): Engineering vector problems There should be a period of practice of a rage of set engineering problems involving the use of the above, until the learner feels confident in solving them. Subjects could include fluid or gas flow, stresses, strains and displacements in structures, and deformation of materials. This may only take a short time it the learner has recently progressed from a lower level of engineering mathematics.	

Lesson 3: Partial differentiation, integration and differential equations **Suggested Teaching Time:** 10 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems
4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources

AC 3.1 Evaluate **partial derivatives** for a function of several variables	Delivery is likely to consist of more whole-class teaching than in the previous lessons, interspersed with self-study, practising problem solving.
	
	Session 1 (10 hours): Uses of partial differentiation
	Learners should be clear about the important uses of partial differentiation:
	• First-order, for example: determining the volume of different shapes; analysing fluid behaviour, heat flow in a fluid etc.
	• Second-order, for example:
	  ○ Analysing the dynamics of rigid bodies, determination of forces and strength of materials
	  ○ Estimating errors in calculated quantities that depend on more than one uncertain experimental
	  ○ Thermodynamic energy functions
	  ○ Expressing most thermodynamic quantities
	The principle of differentiating a function which has two variables should be stressed, i.e.: one of the variables ***a*** is kept constant whilst finding the differential coefficient of the other variable ***b*** , with respect to a, this differential coefficient being the a **partial derivative** of the function
	
	Notation: the symbol ∂ (not a Greek letter but a special symbol) should be used to denote a differential coefficient in an expression with more than one variable. It is important for learners to grasp the meaning of, for example for a familiar equation such as $V = \pi r^2 h$, $\frac{\partial V}{\partial h}$ means **the partial derivative of V with respect to r with h remaining constant.**
		Book:
		
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Lesson 3: Partial differentiation, integration and differential equations **Suggested Teaching Time:** 10 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems

4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Both first- and second-order partial differentiation should be taught using a range of the many engineering examples available and learners should be given further engineering-based problems to practise with. These will include: • Finding the **total differential** for variables which will be changing at the same time • The **chain rule (function of a function)** for partial derivatives: $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$ Differentiate $y = (4x^3 - 5)^4$ Let $(4x^3 - 5) = u$ then **y = u^4** $$\frac{du}{dx} = 12x^2 \text{ and } \frac{dy}{du} = 4u^3$$ *then* $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3 \cdot 12x^2$$ $= 48x^2(4x^3 - 5)^3$ An easier method is to differentiate the bracket, treating it as x^n then differentiate the function inside the bracket. To obtain $\frac{dy}{dx}$ multiply the two results together. Check back to the answer above.	

Lesson 3: Partial differentiation, integration and differential equations **Suggested Teaching Time:** 10 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems

4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
AC 4.1 Evaluate the inaccuracy of calculations that use **approximate numbers**	Differentiate different types e.g. $y = \sqrt{4x^3 + 5x - 4}$ $y = \dfrac{3}{(4t^3 - 7)^5}$ • Rates of change • Identify and evaluate small changes, including the evaluation of errors in calculations that use approximate numbers, by using **partial derivatives**. Approximate numbers are defined as those resulting from limiting the number of decimal places, rounding up or down, and those where significant figures are limited. Time should be spent practising the use of this important method (AC 4.1) which could involve the analysis of actual experimental data.	

Lesson 4: Series and transforms **Suggested Teaching Time:** 22 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems

4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
AC 3.2 Use **series expansions** to obtain approximations of a function	Delivery is likely to be by continued whole class teaching mixed with self study and practice in problem solving. The syllabus could be taught in one of two sequences - following the ACs in order, or each of the subjects could be followed through by combining the ACs in an appropriate sequence. The following is a version of the latter. **Session 1 (4 hours): Series expansions: Maclaurin and Taylor** The first part of the lesson should focus on **Maclaurin's** and **Taylor's** series, beginning with the derivation of Maclaurin: From the power series of f(x) assumed as $\boldsymbol{f(x) = a_0 + a_1 x + a_2 x_2 + a3\, x_3 + a_4 x_4} + \ldots\ldots$ Where $\boldsymbol{a_0, a_1, a_2}$**.........** are constants Arriving at: $$\boldsymbol{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0)} + \cdots$$ The conditions of the Maclaurin's series should also be covered, as should numerical integration limiting values and L'Hôpital's rule for functional evaluation: $$\boldsymbol{\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)}$$ provided $\boldsymbol{g'(a) \neq 0}$ Typical engineering problems using the above should be practised by learners; these are likely to include problems on beam flexure and the derivation of the elastic curve of a transversely loaded beam.	**Book:** Bird. J.O., *Higher Engineering Mathematics* 7th edition (Routledge 2014) ISBN-13: 978-0415662826 **Websites:** http://mathworld.wolfram.com/ http://www.mathcentre.ac.uk/

Lesson 4: Series and transforms

Suggested Teaching Time: 22 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems

4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
    AC 4.2 Use numerical **iterative methods** to find the roots of a function	Taylor's series should then be derived from Maclaurin's series: $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots..$$ Learners should be able to use series expansions to obtain approximations of the functions e^x, $\sin x$, $\cos x$ *and* $\tan x$. **Session 2 (4 hours): Iterative methods** To carry on a logical sequence, the learner should progress to numerical iterative methods of solving equations, starting with the bisection method from first principles and the use of tabular data to find the roots. Secondly the Secant method should be taught : $$x_n = x_{n-1} - f(x_{n-1})\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} = \frac{x_{n-2}f(x_{n-1}) - x_{n-1}\, f(x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}$$ Using initial values x_0 and x_1 close to the root. Third to be introduced is the **Newton-Raphson**, or **Newton Method**: *If* r_1 *is the approximate value of a real root of the equation* $f(x) = 0$ *then a closer approximation to the root* r_2 *is given by:* $$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$	

Lesson 4: Series and transforms **Suggested Teaching Time:** 22 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems

4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
AC 3.3 Obtain a **Fourier series** description for functions of a single variable	Problems in an engineering context should be worked through to consolidate the learning. Subjects could include analysis of waveforms, oscillations and vibrations in beams and other structural members. **Session 3 (4 hours): Fourier series** The following **Fourier Series** should be explored: • sine series • cosine series • half-range sine and cosine series • defined over any finite interval Learners should be proficient in recognising 'odd' and 'even' functions i.e.: • A function is said to be **odd** if $f(-x) = -f(x)$ for all values of x - the graph being symmetrical about the origin • A function is said to be **even** if $f(-x) = f(x)$ for all values of x - the graph being symmetrical about the y axis This will assist greatly in producing the Fourier sine and cosine series for periodic functions Finally the learner should progress to the expansion of a periodic function of period L i.e.: Finding a Fourier series for the function $f(x)$ in the range $-\frac{L}{2} \leq x \geq \frac{L}{2}$ using an additional variable u etc.	

Lesson 4: Series and transforms **Suggested Teaching Time:** 22 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems

4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
AC 3.4 Obtain Laplace transforms for simple **functions**	Contexts for practice problems could include vibration, optics or any system subject to a sinusoidal signal. **Session 4 (5 hours): Laplace transforms** **Laplace transforms** should now be introduced as an alternative solution for solving differential linear equations. Learners should be able to give a definition of the Laplace transform of the function $f(t)$, eg: the integral $\int_0^{\infty} e^{-st} f(t)dt$ where s is a parameter assumed to be a real number. Notations for the Laplace transform of $f(t)$ should be clearly understood and remembered by learners e.g.: • $\mathcal{L}\{f(t)\}$ or $L\ \{f(t)\}$ • $\mathcal{L}\ (f)$ or $L\ f$ • $\bar{f}(s)$ or $f(s)$ Standard Laplace transforms should be covered and learners should be advised to commit some of the basic ones to memory, including : • $f(t) = 1$ • $f(t) = k$ • $f(t) = e^{at}$ • $f(t) = \cos at$ • $f(t) = t$ • $f(t) = t^n$ • $f(t) = \sinh at$	

Lesson 4: Series and transforms **Suggested Teaching Time:** 22 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems

4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
AC 3.5 Obtain the inverse Laplace transforms for simple **functions**	Providing the learners with a list of elementary standard Laplace transforms will be useful e.g.: http://www.mathcentre.ac.uk/resources/uploaded/43799-maths-centre-more-ff-for-web.pdf which also gives data for other topics in this unit **Session 5 (5 hours): Inverse Laplace transforms** Learners should spend some time on worked examples of Laplace transforms before moving on to inverse Laplace transforms. Again, time should be spent on worked examples of **inverse Laplace transforms** involving simple functions until learners are confident and proficient, remembering that these are tools with which they should become very familiar. Learners must be able to obtain **inverse Laplace transforms** for: • Algebraic functions between the limits $t = 0$ and $s = \infty$ and • Trigonometric functions between limits $-\pi$ and π. Finally on this topic the learner should apply the Laplace transform to the **Unit Step** or **Heaviside function**, and the basic theory and solution of the **Dirac delta** function/distribution. These deal with sudden inputs to a system of indefinite duration, and the characteristics of a sudden 'spike' input. Worked examples from different engineering concepts such as electrical switching, impacts, sudden accelerations, the application of the brakes of a moving vehicle and the sudden acceleration and deceleration of a rotating shaft such as in a vehicle transmission.	

Lesson 5: Numerical methods of integration and solving differential equations **Suggested Teaching Time:** 21 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems
4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
AC 3.6 Obtain **integrals** of simple functions	**Session 1 (4 hours): Integrals of simple functions** Delivery, once again, is likely to consist of whole-class teaching interspersed with self-study to practise problem solving. However as learners become more proficient at using the techniques from this and other lessons, they could be divided into groups and be given more complex engineering problems to solve as a team. Returning to calculus, the final lesson in the unit starts with the integration of simple functions. Learners are likely to have worked with integration previously, however it will do no harm at all to revise the basics. The idea of integration as the reverse of differentiation, and also of a process of summation are important and the learner be familiar with the basic rules of these processes. The meaning of the notation should be clear also e.g.: $\int 2x\,dx + c$ means 'the integral of $2x$ with respect to x, and c denotes the possible presence of a constant. Also the standard integrals for common functions should be studied and as many as possible committed to memory, particularly the integrals of a constant, $$ax^n\ (where\ n \neq -1,\ \ \frac{1}{x},\ \ \sin(ax \pm b),\ \ \cos(ax \pm b)\ and\ e^{(ax \pm b)}$$ The concept of limits and evaluating the integral between them should be learned and practised with all of the **standard functions**. The learner should also clearly understand the difference between an **indefinite integral** i.e. one containing the constant 'c' and a **definite integral** i.e. one in which limits have been applied. At this stage, engineering context can be introduced; the learner should be able to solve problems involving, for instance, entropy change, volume of gas or liquid in a container, or voltage waveforms. Learners should be able to find the mean and Root Mean Square (RMS) values of a waveform.	**Book:** Bird. J. O., *Higher Engineering Mathematics* 7th edition (Routledge 2014) ISBN-13: 978-0415662826 **Websites:** http://mathworld.wolfram.com/ http://www.mathcentre.ac.uk/

Lesson 5: Numerical methods of integration and solving differential equations **Suggested Teaching Time:** 21 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems

4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
AC 4.3 Apply **numerical methods of integration** to **engineering variables**	**Session 2 (4 hours): Numerical methods of integration** Numerical methods of integration are important and the learner should be able to apply two rules Trapezoidal rule and Simpson's rule: **The Trapezoidal rule**: for those learners who are unfamiliar with the use of this rule, it is best to start with the arithmetical method of finding the area under an irregular curve. Dividing the area between the curve and the x - y axes into a number of thin strips, each of which approximates to a trapezoid. The area bounded by the curve and the x - y axes is given by : ***(Interval width) [½ (first + last ordinate) + sum of remaining ordinates]*** Quickly progressing then to numerical integration, the learner will see that the above is equal to $\int_a^b y\,dx$. Learners should be able to write down the standard proof of this from memory, and apply the rule to any problem. This applies also to the related **Mid-ordinate rule**. **Simpson's rule** follows on from the preceding two but is a more accurate approximation; learners should always encouraged to think in terms of errors, how they can be reduced in their calculations and how they affect the calculation if they can't be eliminated. Learners should be able to demonstrate how the rule works and how it is derived, including the fact that it only works with an even number of intervals (odd number of ordinates).	

Lesson 5: Numerical methods of integration and solving differential equations **Suggested Teaching Time:** 21 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems
4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
         AC 3.7 Solve **ordinary differential equations**	Engineering-based problems should be practised by the learners, in a range of contexts, which could include fluid flow rates, vibration, distortion of a body or structure under its own weight, spread of a contaminant in a water course or fluid system. In fact any physical property where data are only available at discrete points or intervals. Work with differential equations may be revision for the members of the class, but, as always, there will be those who have not remembered the rules and methods as well as the others. So it is often helpful to revise the families of curves, the notation and the basic rules of differentiation and solving differential equations by separation of variables. **Session 3 (4 hours): First-order ordinary differential equations (AC 3.7 & 4.4)** Learners should be able to recognise and to readily solve first-order differential equations of the following forms: $$\frac{dy}{dx} = f(x)$$ $$\frac{dy}{dx} = f(y)$$ $$\frac{dy}{dx} = f(x) \cdot f(y)$$ Problems in an engineering context could be introduced here.	

Lesson 5: Numerical methods of integration and solving differential equations **Suggested Teaching Time:** 21 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems

4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
AC 4.4 Apply numerical methods for the solution of ordinary differential equation models of engineering systems	**Solving exact first-order differential equations**: learners should be able to define and solve exact first-order differential equations by partial differentiation, and should also be able to test a given equation for exactness. **Solving linear first-order differential equations**: learners should be able to define and recognise a first-order linear differential equation and to solve, using an integrating factor, equations of the form: $\frac{dy}{dx} + Py = Q$ **where** P **and** Q **are functions of** x **only** Learners should practise solving problems in an engineering context, which could include: equations of motion of particles in a resisting fluid, alternating current, impurities in an engine oil tank etc. **Session 4: Second-order ordinary differential equations (AC 3.7 & 4.4)** Learners should be able to identify the types of engineering problem that can be solved using the methods to be addressed in this part of the lesson. These include: free vibration analysis with mass-spring systems, resonant and non-resonant vibration analysis, mechanical and fluid-induced vibration etc. Learners should be familiar with the D-operator and its use and should be able to identify homogeneous and non-homogeneous differential equations. And should be proficient in solving equations of the form: $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy$	

Lesson 5: Numerical methods of integration and solving differential equations **Suggested Teaching Time:** 21 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems

4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Finding the **complementary function (C.F.)** using the D-operator and the **particular integral (P.I.)** using trial functions. This results in the general solution of the above equation: $y = C.F. \cdot P.I.$ Particular integrals should be a focus for learners and they should be encouraged to commit at least some to memory. Learners should understand and be able to apply the procedure for solving ordinary differential equations using **Laplace transforms.** Practice examples should include those involving initial and boundary values, those involving sinusoidal inputs (i.e.: the response of a system to sinusoidal inputs), and those involving resonance.	
AC 4.5 Apply iterative numerical methods to the solution of partial differential equation models of engineering systems	**Session 5 (5 hours): Iterative numerical methods** The final part of the unit involves numerical methods for solving ordinary and partial differential equation models of engineering systems. Examples should cover a range of relevant systems. Learners should approach each method one by one, where appropriate comparing the results and procedures for individual equations. Starting with **ordinary differential equations**, the contextual problems should centre particularly around areas, volumes, centres of gravity and moments of inertia. Methods to be addressed by the learners are: • **Euler**: taking the first two terms of the Taylor series: $f(a+h) = f(a) + hf'(a)$	

Lesson 5: Numerical methods of integration and solving differential equations **Suggested Teaching Time:** 21 hours

Learning Outcome: 3. Be able to use calculus to solve engineering problems

4. Be able to apply numerical analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	• **Euler-Cauchy**: learners should be able to demonstrate the difference in accuracy between this and the standard Euler method • **Taylor series** • **Runge-Kutta**: learners should not be expected to derive or prove this method, however they should be able to follow the seven steps in order to arrive at a highly accurate solution to an equation. Finite difference: forward, backward and central It would be a useful exercise for learners to construct a table of solutions for a particular equation, say $y = x + 1 + e^{x}$ using each of the above methods to compare their accuracies. This will help the learner to remember the relative accuracies and to be able to select an appropriate method for a particular problem. The final subject in this unit is the solution of **partial differential equation models of engineering systems** using **iterative numerical methods.** The following should be included: • **Finite difference** methods (similar to above) • The **Jacobi**[1] iterative method for a square set of linear equations: **Ax = b** • The **Gauss-Seidel**[1] iterative method for a square set of linear equations**: Ax = b** Learners should then complete a series of problems based on typical and relevant engineering systems. The aim is to consolidate all of the unit learning and to give learners practice in selecting and using methods taught in the unit.	

[1] These two methods do not appear in the recommended textbook, but can be found on the Wolfram.com website