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9209-501 NOVEMBER 2015
Level 5 Advanced Technician Diploma in Electrical and Electronic Engineering

Mathematical Formula booklet

Candidates should refer to this 501 formula booklet throughout the course and become familiar with it. Candidates will need to use a clean copy of this document for the 9209-501 exam and for the sample questions.

The online version of the full Mathematical handbook can be found at http://homepage.ntu.edu.tw/~wttsai/MathModel/Mathematical%20Formula%20 Handbook.pdf

(Please note we have extracted formula which is not relevant to this unit. The full online version is not required.)

PRACTICE

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Contents

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1 Series

Arithmetic and Geometric progressions

A.P. $S_n = a + (a+d) + (a+2d) + \cdots + [a + (n-1)d] = \dfrac{n}{2}[2a + (n-1)d]$

G.P. $S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\dfrac{1-r^n}{1-r}$, $\left(S_\infty = \dfrac{a}{1-r} \quad \text{for } |r| < 1 \right)$

(These results also hold for complex series.)

Binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

If n is a positive integer the series terminates and is valid for all x: the term in x^r is ${}^nC_r x^r$ or $\binom{n}{r}$ where ${}^nC_r \equiv \dfrac{n!}{r!(n-r)!}$ is the number of different ways in which an unordered sample of r objects can be selected from a set of n objects without replacement. When n is not a positive integer, the series does not terminate: the infinite series is convergent for $|x| < 1$.

Taylor and Maclaurin Series

If $y(x)$ is well-behaved in the vicinity of $x = a$ then it has a Taylor series,

$$y(x) = y(a+u) = y(a) + u\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{u^2}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{u^3}{3!}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \cdots$$

where $u = x - a$ and the differential coefficients are evaluated at $x = a$. A Maclaurin series is a Taylor series with $a = 0$,

$$y(x) = y(0) + x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x^2}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{x^3}{3!}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \cdots$$

Power series with real variables

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{valid for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \qquad \text{valid for } -1 < x \le 1$$

$$\cos x = \frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{valid for all values of } x$$

$$\sin x = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{valid for all values of } x$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \qquad \text{valid for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad \text{valid for } -1 \le x \le 1$$

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots \qquad \text{valid for } -1 < x < 1$$

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Integer series

$$\sum_1^N n = 1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$\sum_1^N n^2 = 1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_1^N n^3 = 1^3 + 2^3 + 3^3 + \cdots + N^3 = [1 + 2 + 3 + \cdots N]^2 = \frac{N^2(N+1)^2}{4}$$

$$\sum_1^\infty \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2 \qquad \text{[see expansion of } \ln(1+x)\text{]}$$

$$\sum_1^\infty \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \qquad \text{[see expansion of } \tan^{-1} x\text{]}$$

$$\sum_1^\infty \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

$$\sum_1^N n(n+1)(n+2) = 1.2.3 + 2.3.4 + \cdots + N(N+1)(N+2) = \frac{N(N+1)(N+2)(N+3)}{4}$$

This last result is a special case of the more general formula,

$$\sum_1^N n(n+1)(n+2)\ldots(n+r) = \frac{N(N+1)(N+2)\ldots(N+r)(N+r+1)}{r+2}.$$

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2 Vector Algebra

If $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are orthonormal vectors and $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ then $|\boldsymbol{A}|^2 = A_x^2 + A_y^2 + A_z^2$. [Orthonormal vectors $\equiv$ orthogonal unit vectors.]

Scalar product

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{B} &= |\boldsymbol{A}|\,|\boldsymbol{B}|\cos\theta \\ &= A_xB_x + A_yB_y + A_zB_z = [\,A_xA_yA_z\,]\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} \end{aligned}$$

where θ is the angle between the vectors

Scalar multiplication is commutative: $\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{A}$.

Equation of a line

A point $\boldsymbol{r} \equiv (x, y, z)$ lies on a line passing through a point $\boldsymbol{a}$ and parallel to vector $\boldsymbol{b}$ if

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$$

with λ a real number.

Equation of a plane

A point $\boldsymbol{r} \equiv (x, y, z)$ is on a plane if either

(a) $\boldsymbol{r}\cdot\widehat{\boldsymbol{d}} = |\boldsymbol{d}|$, where $\boldsymbol{d}$ is the normal from the origin to the plane, or

(b) $\dfrac{x}{X} + \dfrac{y}{Y} + \dfrac{z}{Z} = 1$ where X, Y, Z are the intercepts on the axes.

Vector product

$\boldsymbol{A}\times\boldsymbol{B} = \boldsymbol{n}\,|\boldsymbol{A}|\,|\boldsymbol{B}|\sin\theta$, where θ is the angle between the vectors and $\boldsymbol{n}$ is a unit vector normal to the plane containing $\boldsymbol{A}$ and $\boldsymbol{B}$ in the direction for which $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{n}$ form a right-handed set of axes.

$\boldsymbol{A}\times\boldsymbol{B}$ in determinant form

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$\boldsymbol{A}\times\boldsymbol{B}$ in matrix form

$$\begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix}\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Vector multiplication is not commutative: $\boldsymbol{A}\times\boldsymbol{B} = -\boldsymbol{B}\times\boldsymbol{A}$.

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3 Matrix Algebra

Unit matrices

The unit matrix I of order n is a square matrix with all diagonal elements equal to one and all off-diagonal elements zero, i.e., $(I)_{ij} = \delta_{ij}$. If A is a square matrix of order n, then $AI = IA = A$. Also $I = I^{-1}$.
I is sometimes written as I_n if the order needs to be stated explicitly.

Products

If A is a $(n \times l)$ matrix and B is a $(l \times m)$ then the product AB is defined by

$$(AB)_{ij} = \sum_{k=1}^{l} A_{ik}B_{kj}$$

In general $AB \neq BA$.

Transpose matrices

If A is a matrix, then transpose matrix A^T is such that $(A^T)_{ij} = (A)_{ji}$.

Inverse matrices

If A is a square matrix with non-zero determinant, then its inverse A^{-1} is such that $AA^{-1} = A^{-1}A = I$.

$$(A^{-1})_{ij} = \frac{\text{transpose of cofactor of } A_{ij}}{|A|}$$

where the cofactor of A_{ij} is $(-1)^{i+j}$ times the determinant of the matrix A with the j-th row and i-th column deleted.

Determinants

If A is a square matrix then the determinant of A, $|A|$ ($\equiv \det A$) is defined by

$$|A| = \sum_{i,j,k,\ldots} \epsilon_{ijk\ldots} A_{1i}A_{2j}A_{3k}\ldots$$

where the number of the suffixes is equal to the order of the matrix.

2×2 matrices

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then,

$$|A| = ad - bc \qquad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Product rules

$$(AB\ldots N)^T = N^T \ldots B^T A^T$$

$$(AB\ldots N)^{-1} = N^{-1} \ldots B^{-1} A^{-1} \qquad \text{(if individual inverses exist)}$$

$$|AB\ldots N| = |A|\,|B|\ldots|N| \qquad \text{(if individual matrices are square)}$$

Orthogonal matrices

An orthogonal matrix Q is a square matrix whose columns q_i form a set of orthonormal vectors. For any orthogonal matrix Q,

$$Q^{-1} = Q^T, \quad |Q| = \pm 1, \quad Q^T \text{ is also orthogonal.}$$

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Solving sets of linear simultaneous equations

If A is square then $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ if A^{-1} exists, i.e., if $|A| \neq 0$.

If A is square then $A\boldsymbol{x} = 0$ has a non-trivial solution if and only if $|A| = 0$.

An over-constrained set of equations $A\boldsymbol{x} = \boldsymbol{b}$ is one in which A has m rows and n columns, where m (the number of equations) is greater than n (the number of variables). The best solution x (in the sense that it minimizes the error $|A\boldsymbol{x} - \boldsymbol{b}|$) is the solution of the n equations $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$. If the columns of A are orthonormal vectors then $\boldsymbol{x} = A^T\boldsymbol{b}$.

Eigenvalues and eigenvectors

The n eigenvalues λ_i and eigenvectors $\boldsymbol{u}_i$ of an $n \times n$ matrix A are the solutions of the equation $A\boldsymbol{u} = \lambda\boldsymbol{u}$. The eigenvalues are the zeros of the polynomial of degree n, $P_n(\lambda) = |A - \lambda I|$. If A is Hermitian then the eigenvalues λ_i are real and the eigenvectors u_i are mutually orthogonal. $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A.

$$\mathrm{Tr}\, A = \sum_i \lambda_i, \quad \text{also } |A| = \prod_i \lambda_i.$$

If S is a symmetric matrix, Λ is the diagonal matrix whose diagonal elements are the eigenvalues of S, and U is the matrix whose columns are the normalized eigenvectors of A, then

$$U^T S U = \Lambda \qquad \text{and} \qquad S = U \Lambda U^T.$$

If x is an approximation to an eigenvector of A then $\boldsymbol{x}^T A\boldsymbol{x}/(\boldsymbol{x}^T\boldsymbol{x})$ (Rayleigh's quotient) is an approximation to the corresponding eigenvalue.

Commutators

$$[A, B] \equiv AB - BA$$

$$[A, B] = -[B, A]$$

$$[A, B]^\dagger = [B^\dagger, A^\dagger]$$

$$[A + B, C] = [A, C] + [B, C]$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

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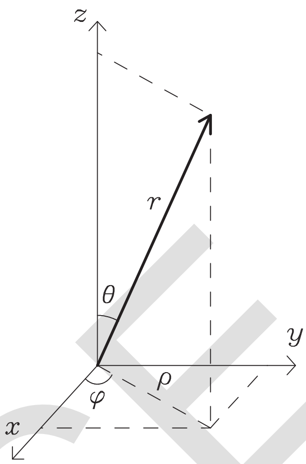
4 Vector Calculus

Notation



ϕ is a scalar function of a set of position coordinates. In Cartesian coordinates $\phi = \phi(x, y, z)$; in cylindrical polar coordinates $\phi = \phi(\rho, \varphi, z)$; in spherical polar coordinates $\phi = \phi(r, \theta, \varphi)$; in cases with radial symmetry $\phi = \phi(r)$. $\boldsymbol{A}$ is a vector function whose components are scalar functions of the position coordinates: in Cartesian coordinates $\boldsymbol{A} = \boldsymbol{i}A_x + \boldsymbol{j}A_y + \boldsymbol{k}A_z$, where A_x, A_y, A_z are independent functions of x, y, z.

In Cartesian coordinates ∇ ('del') $\equiv \boldsymbol{i}\dfrac{\partial}{\partial x} + \boldsymbol{j}\dfrac{\partial}{\partial y} + \boldsymbol{k}\dfrac{\partial}{\partial z} \equiv \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix}$

$$\mathrm{grad}\,\phi = \nabla\phi, \qquad \mathrm{div}\,\boldsymbol{A} = \nabla \cdot \boldsymbol{A}, \qquad \mathrm{curl}\,A = \nabla \times \boldsymbol{A}$$

Identities

$$\mathrm{grad}(\phi_1 + \phi_2) \equiv \mathrm{grad}\,\phi_1 + \mathrm{grad}\,\phi_2 \qquad \mathrm{div}(\boldsymbol{A}_1 + \boldsymbol{A}_2) \equiv \mathrm{div}\,\boldsymbol{A}_1 + \mathrm{div}\,\boldsymbol{A}_2$$

$$\mathrm{grad}(\phi_1\phi_2) \equiv \phi_1\,\mathrm{grad}\,\phi_2 + \phi_2\,\mathrm{grad}\,\phi_1$$

$$\mathrm{curl}(\boldsymbol{A}_{\boldsymbol{1}} + \boldsymbol{A}_{\boldsymbol{2}}) \equiv \mathrm{curl}\,\boldsymbol{A}_1 + \mathrm{curl}\,\boldsymbol{A}_2$$

$$\mathrm{div}(\phi\boldsymbol{A}) \equiv \phi\,\mathrm{div}\,\boldsymbol{A} + (\mathrm{grad}\,\phi) \cdot \boldsymbol{A}, \qquad \mathrm{curl}(\phi\boldsymbol{A}) \equiv \phi\,\mathrm{curl}\,\boldsymbol{A} + (\mathrm{grad}\,\phi) \times \boldsymbol{A}$$

$$\mathrm{div}(\boldsymbol{A}_1 \times \boldsymbol{A}_2) \equiv \boldsymbol{A}_2 \cdot \mathrm{curl}\,\boldsymbol{A}_1 - \boldsymbol{A}_1 \cdot \mathrm{curl}\,\boldsymbol{A}_2$$

$$\mathrm{curl}(\boldsymbol{A}_1 \times \boldsymbol{A}_2) \equiv \boldsymbol{A}_1\,\mathrm{div}\,\boldsymbol{A}_2 - \boldsymbol{A}_2\,\mathrm{div}\,\boldsymbol{A}_1 + (\boldsymbol{A}_2 \cdot \mathrm{grad})\boldsymbol{A}_1 - (\boldsymbol{A}_1 \cdot \mathrm{grad})\boldsymbol{A}_2$$

$$\mathrm{div}(\mathrm{curl}\,\boldsymbol{A}) \equiv 0, \qquad \mathrm{curl}(\mathrm{grad}\,\phi) \equiv 0$$

$$\mathrm{curl}(\mathrm{curl}\,\boldsymbol{A}) \equiv \mathrm{grad}(\mathrm{div}\,\boldsymbol{A}) - \mathrm{div}(\mathrm{grad}\,\boldsymbol{A}) \equiv \mathrm{grad}(\mathrm{div}\,\boldsymbol{A}) - \nabla^2\boldsymbol{A}$$

$$\mathrm{grad}(\boldsymbol{A}_1 \cdot \boldsymbol{A}_2) \equiv \boldsymbol{A}_1 \times (\mathrm{curl}\,\boldsymbol{A}_2) + (\boldsymbol{A}_1 \cdot \mathrm{grad})\boldsymbol{A}_2 + \boldsymbol{A}_2 \times (\mathrm{curl}\,\boldsymbol{A}_1) + (\boldsymbol{A}_2 \cdot \mathrm{grad})\boldsymbol{A}_1$$

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Grad, Div, Curl and the Laplacian

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Cartesian Coordinates		$x = \rho\cos\varphi \quad y = \rho\sin\varphi \quad z = z$	$x = r\cos\varphi\sin\theta \quad y = r\sin\varphi\sin\theta \quad z = r\cos\theta$
Vector $\boldsymbol{A}$	$A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$	$A_\rho\widehat{\boldsymbol{\rho}} + A_\varphi\widehat{\boldsymbol{\varphi}} + A_z\widehat{\boldsymbol{z}}$	$A_r\widehat{\boldsymbol{r}} + A_\theta\widehat{\boldsymbol{\theta}} + A_\varphi\widehat{\boldsymbol{\varphi}}$
Gradient $\nabla\phi$	$\frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$	$\frac{\partial\phi}{\partial\rho}\widehat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\widehat{\boldsymbol{\varphi}} + \frac{\partial\phi}{\partial z}\widehat{\boldsymbol{z}}$	$\frac{\partial\phi}{\partial r}\widehat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\widehat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\widehat{\boldsymbol{\varphi}}$
Divergence $\nabla\cdot\boldsymbol{A}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times\boldsymbol{A}$	$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{\rho}\widehat{\boldsymbol{\rho}} & \widehat{\boldsymbol{\varphi}} & \frac{1}{\rho}\widehat{\boldsymbol{z}} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{r^2\sin\theta}\widehat{\boldsymbol{r}} & \frac{1}{r\sin\theta}\widehat{\boldsymbol{\theta}} & \frac{1}{r}\widehat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian $\nabla^2\phi$	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

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5 Complex Variables

Complex numbers

The complex number $z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{e}^{\mathrm{i}(\theta+2n\pi)}$, where $\mathrm{i}^2 = -1$ and n is an arbitrary integer. The real quantity r is the modulus of z and the angle θ is the argument of z. The complex conjugate of z is $z^* = x - \mathrm{i}y = r(\cos\theta - \mathrm{i}\sin\theta) = r\,\mathrm{e}^{-\mathrm{i}\theta}$; $\quad zz^* = |z|^2 = x^2 + y^2$

De Moivre's theorem

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \mathrm{e}^{\mathrm{i}n\theta} = \cos n\theta + \mathrm{i}\sin n\theta$$

Power series for complex variables.

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \qquad \text{convergent for all finite } z$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad \text{convergent for all finite } z$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \qquad \text{convergent for all finite } z$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad \text{principal value of } \ln(1+z)$$

This last series converges both on and within the circle $|z| = 1$ except at the point $z = -1$.

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

This last series converges both on and within the circle $|z| = 1$ except at the points $z = \pm\mathrm{i}$.

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + \cdots$$

This last series converges both on and within the circle $|z| = 1$ except at the point $z = -1$.

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6 Trigonometric Formulae

$$\cos^2 A + \sin^2 A = 1 \qquad \sec^2 A - \tan^2 A = 1 \qquad \operatorname{cosec}^2 A - \cot^2 A = 1$$

$$\sin 2A = 2\sin A\cos A \qquad \cos 2A = \cos^2 A - \sin^2 A \qquad \tan 2A = \frac{2\tan A}{1-\tan^2 A}.$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B \qquad \cos A\cos B = \frac{\cos(A+B)+\cos(A-B)}{2}$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B \qquad \sin A\sin B = \frac{\cos(A-B)-\cos(A+B)}{2}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B} \qquad \sin A\cos B = \frac{\sin(A+B)+\sin(A-B)}{2}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \qquad \cos^2 A = \frac{1+\cos 2A}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \qquad \sin^2 A = \frac{1-\cos 2A}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \qquad \cos^3 A = \frac{3\cos A+\cos 3A}{4}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2} \qquad \sin^3 A = \frac{3\sin A-\sin 3A}{4}$$

Relations between sides and angles of any plane triangle

In a plane triangle with angles A, B, and C and sides opposite a, b, and c respectively,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter of circumscribed circle.}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a = b\cos C + c\cos B$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

$$\text{area} = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \sqrt{s(s-a)(s-b)(s-c)}, \qquad \text{where } s = \frac{1}{2}(a+b+c)$$

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7 Hyperbolic Functions

$$\cosh x = \frac{1}{2}(\mathrm{e}^{x} + \mathrm{e}^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \text{valid for all } x$$

$$\sinh x = \frac{1}{2}(\mathrm{e}^{x} - \mathrm{e}^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{valid for all } x$$

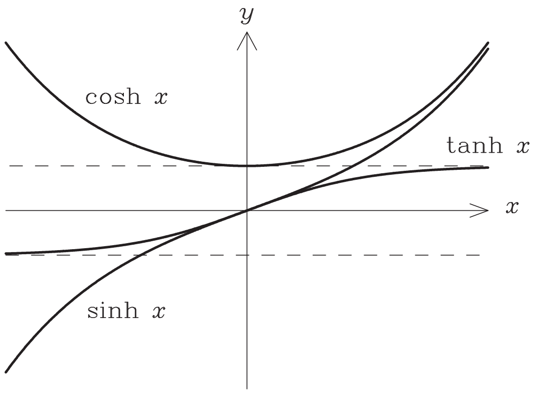
$\cosh \mathrm{i}x = \cos x$ $\qquad$ $\cos \mathrm{i}x = \cosh x$

$\sinh \mathrm{i}x = \mathrm{i}\sin x$ $\qquad$ $\sin \mathrm{i}x = \mathrm{i}\sinh x$

$\tanh x = \dfrac{\sinh x}{\cosh x}$ $\qquad$ $\operatorname{sech} x = \dfrac{1}{\cosh x}$

$\coth x = \dfrac{\cosh x}{\sinh x}$ $\qquad$ $\operatorname{cosech} x = \dfrac{1}{\sinh x}$

$\cosh^2 x - \sinh^2 x = 1$



For large positive x:

$\cosh x \approx \sinh x \to \dfrac{\mathrm{e}^{x}}{2}$

$\tanh x \to 1$

For large negative x:

$\cosh x \approx -\sinh x \to \dfrac{\mathrm{e}^{-x}}{2}$

$\tanh x \to -1$

Relations of the functions

$\sinh x = -\sinh(-x)$ $\qquad$ $\operatorname{sech} x = \operatorname{sech}(-x)$

$\cosh x = \cosh(-x)$ $\qquad$ $\operatorname{cosech} x = -\operatorname{cosech}(-x)$

$\tanh x = -\tanh(-x)$ $\qquad$ $\coth x = -\coth(-x)$

$$\sinh x = \frac{2\tanh(x/2)}{1-\tanh^2(x/2)} = \frac{\tanh x}{\sqrt{1-\tanh^2 x}} \qquad \cosh x = \frac{1+\tanh^2(x/2)}{1-\tanh^2(x/2)} = \frac{1}{\sqrt{1-\tanh^2 x}}$$

$\tanh x = \sqrt{1-\operatorname{sech}^2 x}$ $\qquad$ $\operatorname{sech} x = \sqrt{1-\tanh^2 x}$

$\coth x = \sqrt{\operatorname{cosech}^2 x + 1}$ $\qquad$ $\operatorname{cosech} x = \sqrt{\coth^2 x - 1}$

$$\sinh(x/2) = \sqrt{\frac{\cosh x - 1}{2}} \qquad \cosh(x/2) = \sqrt{\frac{\cosh x + 1}{2}}$$

$$\tanh(x/2) = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$$\sinh(2x) = 2\sinh x\cosh x \qquad \tanh(2x) = \frac{2\tanh x}{1+\tanh^2 x}$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

$$\sinh(3x) = 3\sinh x + 4\sinh^3 x \qquad \cosh 3x = 4\cosh^3 x - 3\cosh x$$

$$\tanh(3x) = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

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$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\sinh x + \sinh y = 2 \sinh \frac{1}{2}(x+y) \cosh \frac{1}{2}(x-y) \qquad \cosh x + \cosh y = 2 \cosh \frac{1}{2}(x+y) \cosh \frac{1}{2}(x-y)$$

$$\sinh x - \sinh y = 2 \cosh \frac{1}{2}(x+y) \sinh \frac{1}{2}(x-y) \qquad \cosh x - \cosh y = 2 \sinh \frac{1}{2}(x+y) \sinh \frac{1}{2}(x-y)$$

$$\sinh x \pm \cosh x = \frac{1 \pm \tanh(x/2)}{1 \mp \tanh(x/2)} = e^{\pm x}$$

$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}$$

$$\coth x \pm \coth y = \pm \frac{\sinh(x \pm y)}{\sinh x \sinh y}$$

Inverse functions

$$\sinh^{-1} \frac{x}{a} = \ln\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right) \qquad \text{for } -\infty < x < \infty$$

$$\cosh^{-1} \frac{x}{a} = \ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) \qquad \text{for } x \geq a$$

$$\tanh^{-1} \frac{x}{a} = \frac{1}{2} \ln\left(\frac{a+x}{a-x}\right) \qquad \text{for } x^2 < a^2$$

$$\coth^{-1} \frac{x}{a} = \frac{1}{2} \ln\left(\frac{x+a}{x-a}\right) \qquad \text{for } x^2 > a^2$$

$$\operatorname{sech}^{-1} \frac{x}{a} = \ln\left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2} - 1}\right) \qquad \text{for } 0 < x \leq a$$

$$\operatorname{cosech}^{-1} \frac{x}{a} = \ln\left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2} + 1}\right) \qquad \text{for } x \neq 0$$

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8 Limits

$n^c x^n \to 0$ as $n \to \infty$ if $|x| < 1$ (any fixed c)

$x^n/n! \to 0$ as $n \to \infty$ (any fixed x)

$(1 + x/n)^n \to \mathrm{e}^x$ as $n \to \infty$, $x \ln x \to 0$ as $x \to 0$

If $f(a) = g(a) = 0$ then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$ (l'Hôpital's rule)

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9 Differentiation

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \cdots + {}^nC_r u^{(n-r)}v^{(r)} + \cdots + uv^{(n)}$$ Leibniz Theorem

where $^nC_r \equiv \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \cos x & \frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) &= \cosh x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) &= -\sin x & \frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) &= \sinh x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) &= \sec^2 x & \frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) &= \operatorname{sech}^2 x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) &= \sec x \tan x & \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} x) &= -\operatorname{sech} x \tanh x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) &= -\operatorname{cosec}^2 x & \frac{\mathrm{d}}{\mathrm{d}x}(\coth x) &= -\operatorname{cosech}^2 x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) &= -\operatorname{cosec} x \cot x & \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosech} x) &= -\operatorname{cosech} x \coth x
\end{aligned}$$

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10 Integration

Standard forms

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + c \qquad \text{for } n \neq -1$$

$$\int \frac{1}{x} \, \mathrm{d}x = \ln x + c \qquad \int \ln x \, \mathrm{d}x = x(\ln x - 1) + c$$

$$\int \mathrm{e}^{ax} \, \mathrm{d}x = \frac{1}{a} \mathrm{e}^{ax} + c \qquad \int x \, \mathrm{e}^{ax} \, \mathrm{d}x = \mathrm{e}^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right) + c$$

$$\int x \ln x \, \mathrm{d}x = \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + c$$

$$\int \frac{1}{a^2 + x^2} \, \mathrm{d}x = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\int \frac{1}{a^2 - x^2} \, \mathrm{d}x = \frac{1}{a} \tanh^{-1} \left(\frac{x}{a} \right) + c = \frac{1}{2a} \ln \left(\frac{a+x}{a-x} \right) + c \qquad \text{for } x^2 < a^2$$

$$\int \frac{1}{x^2 - a^2} \, \mathrm{d}x = -\frac{1}{a} \coth^{-1} \left(\frac{x}{a} \right) + c = \frac{1}{2a} \ln \left(\frac{x-a}{x+a} \right) + c \qquad \text{for } x^2 > a^2$$

$$\int \frac{x}{(x^2 \pm a^2)^n} \, \mathrm{d}x = \frac{-1}{2(n-1)} \frac{1}{(x^2 \pm a^2)^{n-1}} + c \qquad \text{for } n \neq 1$$

$$\int \frac{x}{x^2 \pm a^2} \, \mathrm{d}x = \frac{1}{2} \ln(x^2 \pm a^2) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x = \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} \, \mathrm{d}x = \ln \left(x + \sqrt{x^2 \pm a^2} \right) + c$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} \, \mathrm{d}x = \sqrt{x^2 \pm a^2} + c$$

$$\int \sqrt{a^2 - x^2} \, \mathrm{d}x = \frac{1}{2} \left[x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \left(\frac{x}{a} \right) \right] + c$$

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$$\int_0^\infty \frac{1}{(1+x)x^p}\,\mathrm{d}x = \pi \operatorname{cosec} p\pi \qquad \text{for } p<1$$

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\int_{-\infty}^{\infty} \exp(-x^2/2\sigma^2)\,\mathrm{d}x = \sigma\sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} x^n \exp(-x^2/2\sigma^2)\,\mathrm{d}x = \begin{cases} 1\times 3\times 5\times\cdots(n-1)\sigma^{n+1}\sqrt{2\pi} & \text{for } n\geq 2 \text{ and even} \\ 0 & \text{for } n\geq 1 \text{ and odd} \end{cases}$$

$$\int \sin x\,\mathrm{d}x = -\cos x + c \qquad \int \sinh x\,\mathrm{d}x = \cosh x + c$$

$$\int \cos x\,\mathrm{d}x = \sin x + c \qquad \int \cosh x\,\mathrm{d}x = \sinh x + c$$

$$\int \tan x\,\mathrm{d}x = -\ln(\cos x) + c \qquad \int \tanh x\,\mathrm{d}x = \ln(\cosh x) + c$$

$$\int \operatorname{cosec} x\,\mathrm{d}x = \ln(\operatorname{cosec} x - \cot x) + c \qquad \int \operatorname{cosech} x\,\mathrm{d}x = \ln\left[\tanh(x/2)\right] + c$$

$$\int \sec x\,\mathrm{d}x = \ln(\sec x + \tan x) + c \qquad \int \operatorname{sech} x\,\mathrm{d}x = 2\tan^{-1}(\mathrm{e}^x) + c$$

$$\int \cot x\,\mathrm{d}x = \ln(\sin x) + c \qquad \int \coth x\,\mathrm{d}x = \ln(\sinh x) + c$$

$$\int \sin mx \sin nx\,\mathrm{d}x = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c \qquad \text{if } m^2 \neq n^2$$

$$\int \cos mx \cos nx\,\mathrm{d}x = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + c \qquad \text{if } m^2 \neq n^2$$

Standard substitutions

If the integrand is a function of:	substitute:
(a^2-x^2) or $\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
(x^2+a^2) or $\sqrt{x^2+a^2}$	$x = a\tan\theta$ or $x = a\sinh\theta$
(x^2-a^2) or $\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $x = a\cosh\theta$

If the integrand is a rational function of $\sin x$ or $\cos x$ or both, substitute $t = \tan(x/2)$ and use the results:

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2}.$$

If the integrand is of the form:	substitute:
$\displaystyle\int \frac{\mathrm{d}x}{(ax+b)\sqrt{px+q}}$	$px+q = u^2$
$\displaystyle\int \frac{\mathrm{d}x}{(ax+b)\sqrt{px^2+qx+r}}$	$ax+b = \dfrac{1}{u}.$

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Integration by parts

$$\int_a^b u \, \mathrm{d}v = uv\Big|_a^b - \int_a^b v \, \mathrm{d}u$$

Differentiation of an integral

If $f(x, \alpha)$ is a function of x containing a parameter α and the limits of integration a and b are functions of α then

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha) \, \mathrm{d}x = f(b,\alpha)\frac{\mathrm{d}b}{\mathrm{d}\alpha} - f(a,\alpha)\frac{\mathrm{d}a}{\mathrm{d}\alpha} + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x,\alpha) \, \mathrm{d}x.$$

Special case,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(y) \, \mathrm{d}y = f(x).$$

Dirac δ-'function'

$$\delta(t-\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[\mathrm{i}\omega(t-\tau)] \, \mathrm{d}\omega.$$

If $f(t)$ is an arbitrary function of t then $\int_{-\infty}^{\infty} \delta(t-\tau) f(t) \, \mathrm{d}t = f(\tau)$.

$\delta(t) = 0$ if $t \neq 0$, also $\int_{-\infty}^{\infty} \delta(t) \, \mathrm{d}t = 1$

Reduction formulae

Factorials

$n! = n(n-1)(n-2)\ldots1, \qquad 0! = 1.$

Stirling's formula for large n: $\ln(n!) \approx n \ln n - n$.

For any $p > -1$, $\int_0^{\infty} x^p \, \mathrm{e}^{-x} \, \mathrm{d}x = p \int_0^{\infty} x^{p-1} \, \mathrm{e}^{-x} \, \mathrm{d}x = p!$. $\qquad (-1/2)! = \sqrt{\pi}, \qquad (1/2)! = \sqrt{\pi}/2$, etc.

For any $p, q > -1$, $\int_0^1 x^p (1-x)^q \, \mathrm{d}x = \dfrac{p!q!}{(p+q+1)!}$.

Trigonometrical

If m, n are integers,

$$\int_0^{\pi/2} \sin^m\theta \, \cos^n\theta \, \mathrm{d}\theta = \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2}\theta \, \cos^n\theta \, \mathrm{d}\theta = \frac{n-1}{m+n} \int_0^{\pi/2} \sin^m\theta \, \cos^{n-2}\theta \, \mathrm{d}\theta$$

and can therefore be reduced eventually to one of the following integrals

$$\int_0^{\pi/2} \sin\theta \, \cos\theta \, \mathrm{d}\theta = \frac{1}{2}, \qquad \int_0^{\pi/2} \sin\theta \, \mathrm{d}\theta = 1, \qquad \int_0^{\pi/2} \cos\theta \, \mathrm{d}\theta = 1, \qquad \int_0^{\pi/2} \mathrm{d}\theta = \frac{\pi}{2}.$$

Other

If $I_n = \int_0^{\infty} x^n \exp(-\alpha x^2) \, \mathrm{d}x$ then $I_n = \dfrac{(n-1)}{2\alpha} I_{n-2}, \quad I_0 = \dfrac{1}{2}\sqrt{\dfrac{\pi}{\alpha}}, \quad I_1 = \dfrac{1}{2\alpha}$.

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11 Differential Equations

Diffusion (conduction) equation

$$\frac{\partial \psi}{\partial t} = \kappa \nabla^2 \psi$$

Wave equation

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

Bessel's equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - m^2) y = 0,$$

solutions of which are Bessel functions $J_m(x)$ of order m.

Series form of Bessel functions of the first kind

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{m+2k}}{k!(m+k)!} \qquad \text{(integer } m\text{)}.$$

The same general form holds for non-integer $m > 0$.

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Laplace's equation

$$\nabla^2 u = 0$$

If expressed in two-dimensional polar coordinates (see section 4), a solution is

$$u(\rho, \varphi) = \left[A\rho^n + B\rho^{-n}\right]\left[C\exp(in\varphi) + D\exp(-in\varphi)\right]$$

where A, B, C, D are constants and n is a real integer.

If expressed in three-dimensional polar coordinates (see section 4) a solution is

$$u(r, \theta, \varphi) = \left[Ar^l + Br^{-(l+1)}\right]P_l^m\left[C\sin m\varphi + D\cos m\varphi\right]$$

where l and m are integers with $l \geq |m| \geq 0$; A, B, C, D are constants;

$$P_l^m(\cos\theta) = \sin^{|m|}\theta\left[\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}\right]^{|m|} P_l(\cos\theta)$$

is the associated Legendre polynomial.

$$P_l^0(1) = 1.$$

If expressed in cylindrical polar coordinates (see section 4), a solution is

$$u(\rho, \varphi, z) = J_m(n\rho)\left[A\cos m\varphi + B\sin m\varphi\right]\left[C\exp(nz) + D\exp(-nz)\right]$$

where m and n are integers; A, B, C, D are constants.

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12 Functions of Several Variables

If $\phi = f(x, y, z, \ldots)$ then $\dfrac{\partial \phi}{\partial x}$ implies differentiation with respect to x keeping $y, z, \ldots$ constant.

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{d}z + \cdots \quad \text{and} \quad \delta\phi \approx \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z + \cdots$$

where $x, y, z, \ldots$ are independent variables. $\dfrac{\partial \phi}{\partial x}$ is also written as $\left(\dfrac{\partial \phi}{\partial x}\right)_{y,\ldots}$ or $\left.\dfrac{\partial \phi}{\partial x}\right|_{y,\ldots}$ when the variables kept constant need to be stated explicitly.

If ϕ is a well-behaved function then $\dfrac{\partial^2 \phi}{\partial x\, \partial y} = \dfrac{\partial^2 \phi}{\partial y\, \partial x}$ etc.

If $\phi = f(x, y)$,

$$\left(\frac{\partial \phi}{\partial x}\right)_y = \frac{1}{\left(\dfrac{\partial x}{\partial \phi}\right)_y}, \qquad \left(\frac{\partial \phi}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_\phi \left(\frac{\partial y}{\partial \phi}\right)_x = -1.$$

Taylor series for two variables

If $\phi(x, y)$ is well-behaved in the vicinity of $x = a, y = b$ then it has a Taylor series

$$\phi(x, y) = \phi(a + u, b + v) = \phi(a, b) + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + \frac{1}{2!}\left(u^2\frac{\partial^2 \phi}{\partial x^2} + 2uv\frac{\partial^2 \phi}{\partial x\, \partial y} + v^2\frac{\partial^2 \phi}{\partial y^2}\right) + \cdots$$

where $x = a + u, y = b + v$ and the differential coefficients are evaluated at $x = a, \quad y = b$

Stationary points

A function $\phi = f(x, y)$ has a stationary point when $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \phi}{\partial y} = 0$. Unless $\dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{\partial^2 \phi}{\partial x\, \partial y} = 0$, the following conditions determine whether it is a minimum, a maximum or a saddle point.

$$\left.\begin{array}{lll} \text{Minimum:} & \dfrac{\partial^2 \phi}{\partial x^2} > 0, \text{ or} & \dfrac{\partial^2 \phi}{\partial y^2} > 0, \\ \text{Maximum:} & \dfrac{\partial^2 \phi}{\partial x^2} < 0, \text{ or} & \dfrac{\partial^2 \phi}{\partial y^2} < 0, \end{array}\right\} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} > \left(\frac{\partial^2 \phi}{\partial x\, \partial y}\right)^2$$

$$\text{Saddle point:} \quad \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} < \left(\frac{\partial^2 \phi}{\partial x\, \partial y}\right)^2$$

If $\dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{\partial^2 \phi}{\partial x\, \partial y} = 0$ the character of the turning point is determined by the next higher derivative.

Changing variables: the chain rule

If $\phi = f(x, y, \ldots)$ and the variables $x, y, \ldots$ are functions of independent variables $u, v, \ldots$ then

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + \cdots$$

$$\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial v} + \cdots$$

etc.

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Changing variables in surface and volume integrals – Jacobians

If an area A in the x, y plane maps into an area A' in the u, v plane then

$$\int_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{A'} f(u,v) J\,\mathrm{d}u\,\mathrm{d}v \quad \text{where} \quad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian J is also written as $\dfrac{\partial(x,y)}{\partial(u,v)}$. The corresponding formula for volume integrals is

$$\int_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{V'} f(u,v,w) J\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \qquad \text{where now} \qquad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

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13 Fourier Series and Transforms

Fourier series

If $y(x)$ is a function defined in the range $-\pi \le x \le \pi$ then

$$y(x) \approx c_0 + \sum_{m=1}^{M} c_m \cos mx + \sum_{m=1}^{M'} s_m \sin mx$$

where the coefficients are

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x)\, \mathrm{d}x$$

$$c_m = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos mx\, \mathrm{d}x \qquad (m = 1, \ldots, M)$$

$$s_m = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin mx\, \mathrm{d}x \qquad (m = 1, \ldots, M')$$

with convergence to $y(x)$ as $M, M' \to \infty$ for all points where $y(x)$ is continuous.

Fourier series for other ranges

Variable t, range $0 \le t \le T$, (i.e., a periodic function of time with period T, frequency $\omega = 2\pi/T$).

$$y(t) \approx c_0 + \sum c_m \cos m\omega t + \sum s_m \sin m\omega t$$

where

$$c_0 = \frac{\omega}{2\pi} \int_0^T y(t)\, \mathrm{d}t, \quad c_m = \frac{\omega}{\pi} \int_0^T y(t) \cos m\omega t\, \mathrm{d}t, \quad s_m = \frac{\omega}{\pi} \int_0^T y(t) \sin m\omega t\, \mathrm{d}t.$$

Variable x, range $0 \le x \le L$,

$$y(x) \approx c_0 + \sum c_m \cos \frac{2m\pi x}{L} + \sum s_m \sin \frac{2m\pi x}{L}$$

where

$$c_0 = \frac{1}{L} \int_0^L y(x)\, \mathrm{d}x, \quad c_m = \frac{2}{L} \int_0^L y(x) \cos \frac{2m\pi x}{L}\, \mathrm{d}x, \quad s_m = \frac{2}{L} \int_0^L y(x) \sin \frac{2m\pi x}{L}\, \mathrm{d}x.$$

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Fourier series for odd and even functions

If $y(x)$ is an *odd* (anti-symmetric) function [i.e., $y(-x) = -y(x)$] defined in the range $-\pi \leq x \leq \pi$, then only sines are required in the Fourier series and $s_m = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} y(x)\sin mx \; \mathrm{d}x$. If, in addition, $y(x)$ is symmetric about $x = \pi/2$, then the coefficients s_m are given by $s_m = 0$ (for m even), $s_m = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2} y(x)\sin mx \; \mathrm{d}x$ (for m odd). If $y(x)$ is an *even* (symmetric) function [i.e., $y(-x) = y(x)$] defined in the range $-\pi \leq x \leq \pi$, then only constant and cosine terms are required in the Fourier series and $c_0 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} y(x) \; \mathrm{d}x$, $\quad c_m = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} y(x)\cos mx \; \mathrm{d}x$. If, in addition, $y(x)$ is anti-symmetric about $x = \dfrac{\pi}{2}$, then $c_0 = 0$ and the coefficients c_m are given by $c_m = 0$ (for m even), $c_m = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2} y(x)\cos mx \; \mathrm{d}x$ (for m odd).

[These results also apply to Fourier series with more general ranges provided appropriate changes are made to the limits of integration.]

Complex form of Fourier series

If $y(x)$ is a function defined in the range $-\pi \leq x \leq \pi$ then

$$y(x) \approx \sum_{-M}^{M} C_m \, \mathrm{e}^{\mathrm{i}mx}, \quad C_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(x)\, \mathrm{e}^{-\mathrm{i}mx} \; \mathrm{d}x$$

with m taking all integer values in the range $\pm M$. This approximation converges to $y(x)$ as $M \to \infty$ under the same conditions as the real form.

For other ranges the formulae are:

Variable t, range $0 \leq t \leq T$, frequency $\omega = 2\pi/T$,

$$y(t) = \sum_{-\infty}^{\infty} C_m \, \mathrm{e}^{\mathrm{i}m\omega t}, \quad C_m = \frac{\omega}{2\pi}\int_0^{T} y(t)\, \mathrm{e}^{-\mathrm{i}m\omega t} \; \mathrm{d}t.$$

Variable x', range $0 \leq x' \leq L$,

$$y(x') = \sum_{-\infty}^{\infty} C_m \, \mathrm{e}^{\mathrm{i}2m\pi x'/L}, \quad C_m = \frac{1}{L}\int_0^{L} y(x')\, \mathrm{e}^{-\mathrm{i}2m\pi x'/L} \; \mathrm{d}x'.$$

Discrete Fourier series

If $y(x)$ is a function defined in the range $-\pi \leq x \leq \pi$ which is sampled in the $2N$ equally spaced points $x_n = nx/N \quad [n = -(N-1)\ldots N]$, then

$$\begin{aligned} y(x_n) = {} & c_0 + c_1 \cos x_n + c_2 \cos 2x_n + \cdots + c_{N-1}\cos(N-1)x_n + c_N \cos Nx_n \\ & + s_1 \sin x_n + s_2 \sin 2x_n + \cdots + s_{N-1}\sin(N-1)x_n + s_N \sin Nx_n \end{aligned}$$

where the coefficients are

$$c_0 = \frac{1}{2N}\sum y(x_n)$$

$$c_m = \frac{1}{N}\sum y(x_n)\cos mx_n \qquad (m = 1, \ldots, N-1)$$

$$c_N = \frac{1}{2N}\sum y(x_n)\cos Nx_n$$

$$s_m = \frac{1}{N}\sum y(x_n)\sin mx_n \qquad (m = 1, \ldots, N-1)$$

$$s_N = \frac{1}{2N}\sum y(x_n)\sin Nx_n$$

each summation being over the $2N$ sampling points x_n.

PRACTICE

Fourier transforms

If $y(x)$ is a function defined in the range $-\infty \leq x \leq \infty$ then the Fourier transform $\widehat{y}(\omega)$ is defined by the equations

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{y}(\omega)\, e^{i\omega t}\, d\omega, \qquad \widehat{y}(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-i\omega t}\, dt.$$

If ω is replaced by $2\pi f$, where f is the frequency, this relationship becomes

$$y(t) = \int_{-\infty}^{\infty} \widehat{y}(f)\, e^{i2\pi ft}\, df, \qquad \widehat{y}(f) = \int_{-\infty}^{\infty} y(t)\, e^{-i2\pi ft}\, dt.$$

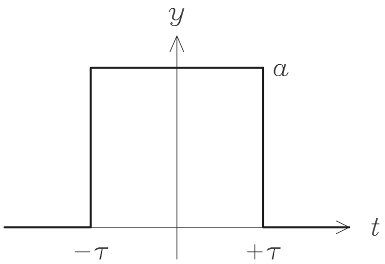
If $y(t)$ is symmetric about $t = 0$ then

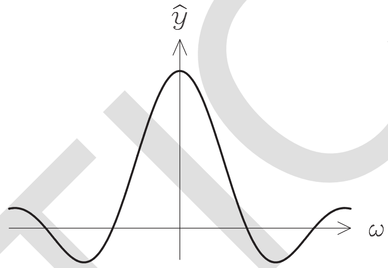
$$y(t) = \frac{1}{\pi}\int_{0}^{\infty} \widehat{y}(\omega) \cos\omega t\, d\omega, \qquad \widehat{y}(\omega) = 2\int_{0}^{\infty} y(t) \cos\omega t\, dt.$$

If $y(t)$ is anti-symmetric about $t = 0$ then

$$y(t) = \frac{1}{\pi}\int_{0}^{\infty} \widehat{y}(\omega) \sin\omega t\, d\omega, \qquad \widehat{y}(\omega) = 2\int_{0}^{\infty} y(t) \sin\omega t\, dt.$$

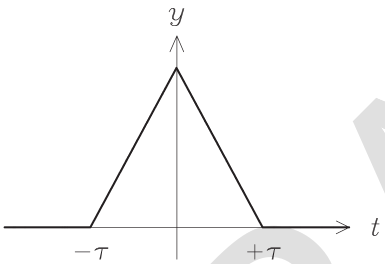
Specific cases

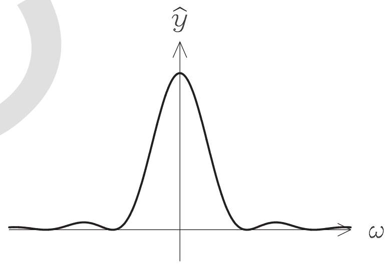




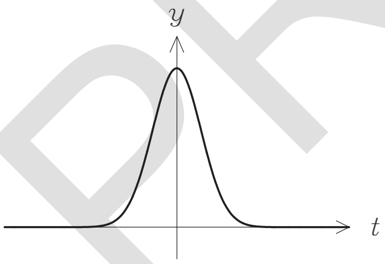
$$\left.\begin{aligned} y(t) &= a, \quad |t| \leq \tau \\ &= 0, \quad |t| > \tau \end{aligned}\right\} \quad \textit{(‘Top Hat’)}, \qquad \widehat{y}(\omega) = 2a\frac{\sin\omega\tau}{\omega} \equiv 2a\tau\,\mathrm{sinc}(\omega\tau)$$

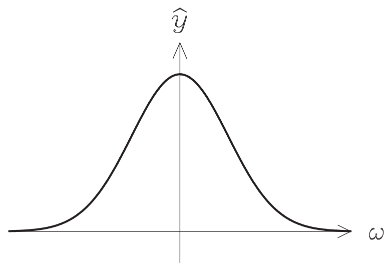
where $\quad \mathrm{sinc}(x) = \dfrac{\sin(x)}{x}$





$$\left.\begin{aligned} y(t) &= a(1 - |t|/\tau), \quad & |t| \leq \tau \\ &= 0, & |t| > \tau \end{aligned}\right\} \quad \textit{(‘Saw-tooth’)}, \qquad \widehat{y}(\omega) = \frac{2a}{\omega^2\tau}(1 - \cos\omega\tau) = a\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2}\right)$$





$$y(t) = \exp(-t^2/t_0^2) \quad \textit{(Gaussian)}, \qquad \widehat{y}(\omega) = t_0\sqrt{\pi}\exp\left(-\omega^2 t_0^2/4\right)$$

$$y(t) = f(t)\, e^{i\omega_0 t} \quad \textit{(modulated function)}, \qquad \widehat{y}(\omega) = \widehat{f}(\omega - \omega_0)$$

$$y(t) = \sum_{m=-\infty}^{\infty} \delta(t - m\tau) \quad \textit{(sampling function)} \qquad \widehat{y}(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n/\tau)$$

PRACTICE

Convolution theorem

If $z(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)\,\mathrm{d}\tau \quad \equiv x(t) * y(t)$ then $\quad \widehat{z}(\omega) = \widehat{x}(\omega)\,\widehat{y}(\omega)$.

Conversely, $\widehat{xy} = \widehat{x} * \widehat{y}$.

PRACTICE

14 Laplace Transforms

If $y(t)$ is a function defined for $t \geq 0$, the Laplace transform $\overline{y}(s)$ is defined by the equation

$$\overline{y}(s) = \mathcal{L}\{y(t)\} = \int_0^\infty \mathrm{e}^{-st} y(t)\,\mathrm{d}t$$

Function $y(t) \quad (t > 0)$	Transform $\overline{y}(s)$	
$\delta(t)$	1	Delta function
$\theta(t)$	$\dfrac{1}{s}$	Unit step function
t^n	$\dfrac{n!}{s^{n+1}}$	
$t^{\frac{1}{2}}$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{s^3}}$	
$t^{-\frac{1}{2}}$	$\sqrt{\dfrac{\pi}{s}}$	
e^{-at}	$\dfrac{1}{(s+a)}$	
$\sin \omega t$	$\dfrac{\omega}{(s^2+\omega^2)}$	
$\cos \omega t$	$\dfrac{s}{(s^2+\omega^2)}$	
$\sinh \omega t$	$\dfrac{\omega}{(s^2-\omega^2)}$	
$\cosh \omega t$	$\dfrac{s}{(s^2-\omega^2)}$	
$\mathrm{e}^{-at} y(t)$	$\overline{y}(s+a)$	
$y(t-\tau)\,\theta(t-\tau)$	$\mathrm{e}^{-s\tau}\,\overline{y}(s)$	
$t y(t)$	$-\dfrac{\mathrm{d}\overline{y}}{\mathrm{d}s}$	
$\dfrac{\mathrm{d}y}{\mathrm{d}t}$	$s\overline{y}(s) - y(0)$	
$\dfrac{\mathrm{d}^n y}{\mathrm{d}t^n}$	$s^n\overline{y}(s) - s^{n-1}y(0) - s^{n-2}\left[\dfrac{\mathrm{d}y}{\mathrm{d}t}\right]_0 \cdots - \left[\dfrac{\mathrm{d}^{n-1}y}{\mathrm{d}t^{n-1}}\right]_0$	
$\displaystyle\int_0^t y(\tau)\,\mathrm{d}\tau$	$\dfrac{\overline{y}(s)}{s}$	
$\left.\begin{array}{l}\displaystyle\int_0^t x(\tau)\,y(t-\tau)\,\mathrm{d}\tau \\ \displaystyle\int_0^t x(t-\tau)\,y(\tau)\,\mathrm{d}\tau\end{array}\right\}$	$\overline{x}(s)\,\overline{y}(s)$	Convolution theorem

[Note that if $y(t) = 0$ for $t < 0$ then the Fourier transform of $y(t)$ is $\widehat{y}(\omega) = \overline{y}(\mathrm{i}\omega)$.]

PRACTICE

15 Numerical Analysis

Finding the zeros of equations

If the equation is $y = f(x)$ and x_n is an approximation to the root then either

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \text{(Newton)}$$

$$\text{or, } x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \qquad \text{(Linear interpolation)}$$

are, in general, better approximations.

Numerical integration of differential equations

If $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ then

$$y_{n+1} = y_n + hf(x_n, y_n) \quad \text{where } h = x_{n+1} - x_n \qquad \text{(Euler method)}$$

Putting $y^*_{n+1} = y_n + hf(x_n, y_n)$ (improved Euler method)

then $y_{n+1} = y_n + \dfrac{h[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]}{2}$

Numerical evaluation of definite integrals

Trapezoidal rule

The interval of integration is divided into n equal sub-intervals, each of width h; then

$$\int_a^b f(x)\,\mathrm{d}x \approx h\left[c\frac{1}{2}f(a) + f(x_1) + \cdots + f(x_j) + \cdots + \frac{1}{2}f(b)\right]$$

where $h = (b-a)/n$ and $x_j = a + jh$.

Simpson's rule

The interval of integration is divided into an even number (say $2n$) of equal sub-intervals, each of width $h = (b-a)/2n$; then

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(b)]$$

Gauss's integration formulae

These have the general form $\displaystyle\int_{-1}^{1} y(x)\,\mathrm{d}x \approx \sum_1^n c_i y(x_i)$

For $n = 2$: $\quad x_i = \pm 0{\cdot}5773; \quad c_i = 1, 1$ (exact for any cubic).

For $n = 3$: $\quad x_i = -0{\cdot}7746, 0{\cdot}0, 0{\cdot}7746; \quad c_i = 0{\cdot}555, 0{\cdot}888, 0{\cdot}555$ (exact for any quintic).

PRACTICE