9209-501 Mathematical Formula booklet

Candidates should refer to this 501 formula booklet throughout the course and become familiar with it. Candidates will need to use a clean copy of this document for the 9205-501 exam and for the sample questions.

The online version of the full Mathematical handbook can be found at
http://homepage.ntu.edu.tw/~wttsai/MathModel/Mathematical%20Formula%20Handbook.pdf

(Please note we have extracted formula which is relevant to this unit. The full online version is not required.)

Contents

Series

Arithmetic and Geometric progressions

A.P. $S_n = a + (a+d) + (a+2d) + \cdots + [a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$

G.P. $S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1-r^n}{1-r}$, $\qquad \left(S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1\right)$

(These results also hold for complex series.)

Binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

If n is a positive integer the series terminates and is valid for all x: the term in x^r is ${}^nC_r x^r$ or $\binom{n}{r}$ where ${}^nC_r \equiv \frac{n!}{r!(n-r)!}$ is the number of different ways in which an unordered sample of r objects can be selected from a set of n objects without replacement. When n is not a positive integer, the series does not terminate: the infinite series is convergent for $|x| < 1$.

Taylor and Maclaurin Series

If $y(x)$ is well-behaved in the vicinity of $x = a$ then it has a Taylor series,

$$y(x) = y(a+u) = y(a) + u\frac{dy}{dx} + \frac{u^2}{2!}\frac{d^2y}{dx^2} + \frac{u^3}{3!}\frac{d^3y}{dx^3} + \cdots$$

where $u = x - a$ and the differential coefficients are evaluated at $x = a$. A Maclaurin series is a Taylor series with $a = 0$,

$$y(x) = y(0) + x\frac{dy}{dx} + \frac{x^2}{2!}\frac{d^2y}{dx^2} + \frac{x^3}{3!}\frac{d^3y}{dx^3} + \cdots$$

Power series with real variables

e^x	$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$	valid for all x
$\ln(1+x)$	$= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$	valid for $-1 < x \le 1$
$\cos x$	$= \frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	valid for all values of x
$\sin x$	$= \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$	valid for all values of x
$\tan x$	$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$	valid for $-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\tan^{-1} x$	$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$	valid for $-1 \le x \le 1$
$\sin^{-1} x$	$= x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots$	valid for $-1 < x < 1$

Integer series

$$\sum_{1}^{N} n = 1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$\sum_{1}^{N} n^2 = 1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{1}^{N} n^3 = 1^3 + 2^3 + 3^3 + \cdots + N^3 = [1 + 2 + 3 + \cdots N]^2 = \frac{N^2(N+1)^2}{4}$$

$$\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$$ [see expansion of $\ln(1+x)$]

$$\sum_{1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$ [see expansion of $\tan^{-1} x$]

$$\sum_{1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

$$\sum_{1}^{N} n(n+1)(n+2) = 1.2.3 + 2.3.4 + \cdots + N(N+1)(N+2) = \frac{N(N+1)(N+2)(N+3)}{4}$$

This last result is a special case of the more general formula,

$$\sum_{1}^{N} n(n+1)(n+2)\ldots(n+r) = \frac{N(N+1)(N+2)\ldots(N+r)(N+r+1)}{r+2}.$$

Vector Algebra

If i, j, k are orthonormal vectors and $A = A_x i + A_y j + A_z k$ then $|A|^2 = A_x^2 + A_y^2 + A_z^2$. [Orthonormal vectors ≡ orthogonal unit vectors.]

Scalar product

$$A \cdot B = |A|\,|B| \cos\theta \qquad \text{where } \theta \text{ is the angle between the vectors}$$

$$= A_x B_x + A_y B_y + A_z B_z = [\,A_x A_y A_z\,] \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Scalar multiplication is commutative: $A \cdot B = B \cdot A$.

Equation of a line

A point $r \equiv (x, y, z)$ lies on a line passing through a point a and parallel to vector b if

$$r = a + \lambda b$$

with λ a real number.

Equation of a plane

A point $r \equiv (x, y, z)$ is on a plane if either

(a) $r \cdot \hat{d} = |d|$, where d is the normal from the origin to the plane, or

(b) $\dfrac{x}{X} + \dfrac{y}{Y} + \dfrac{z}{Z} = 1$ where X, Y, Z are the intercepts on the axes.

Vector product

$A \times B = n\,|A|\,|B| \sin\theta$, where θ is the angle between the vectors and n is a unit vector normal to the plane containing A and B in the direction for which A, B, n form a right-handed set of axes.

$A \times B$ in determinant form

$$\begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$A \times B$ in matrix form

$$\begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Vector multiplication is not commutative: $A \times B = -B \times A$.

Matrix Algebra

Unit matrices

The unit matrix I of order n is a square matrix with all diagonal elements equal to one and all off-diagonal elements zero, i.e., $(I)_{ij} = \delta_{ij}$. If A is a square matrix of order n, then $AI = IA = A$. Also $I = I^{-1}$.
I is sometimes written as I_n if the order needs to be stated explicitly.

Products

If A is a $(n \times l)$ matrix and B is a $(l \times m)$ then the product AB is defined by

$$(AB)_{ij} = \sum_{k=1}^{l} A_{ik} B_{kj}$$

In general $AB \neq BA$.

Transpose matrices

If A is a matrix, then transpose matrix A^T is such that $(A^T)_{ij} = (A)_{ji}$.

Inverse matrices

If A is a square matrix with non-zero determinant, then its inverse A^{-1} is such that $AA^{-1} = A^{-1}A = I$.

$$(A^{-1})_{ij} = \frac{\text{transpose of cofactor of } A_{ij}}{|A|}$$

where the cofactor of A_{ij} is $(-1)^{i+j}$ times the determinant of the matrix A with the j-th row and i-th column deleted.

Determinants

If A is a square matrix then the determinant of A, $|A|$ ($\equiv \det A$) is defined by

$$|A| = \sum_{i,j,k,\ldots} \epsilon_{ijk\ldots} A_{1i} A_{2j} A_{3k} \ldots$$

where the number of the suffixes is equal to the order of the matrix.

2×2 matrices

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then,

$$|A| = ad - bc \qquad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Product rules

$$(AB \ldots N)^T = N^T \ldots B^T A^T$$

$$(AB \ldots N)^{-1} = N^{-1} \ldots B^{-1} A^{-1} \qquad \text{(if individual inverses exist)}$$

$$|AB \ldots N| = |A|\,|B| \ldots |N| \qquad \text{(if individual matrices are square)}$$

Orthogonal matrices

An orthogonal matrix Q is a square matrix whose columns q_i form a set of orthonormal vectors. For any orthogonal matrix Q,

$$Q^{-1} = Q^T, \quad |Q| = \pm 1, \quad Q^T \text{ is also orthogonal.}$$

Solving sets of linear simultaneous equations

If A is square then $Ax = b$ has a unique solution $x = A^{-1}b$ if A^{-1} exists, i.e., if $|A| \neq 0$.
If A is square then $Ax = 0$ has a non-trivial solution if and only if $|A| = 0$.
An over-constrained set of equations $Ax = b$ is one in which A has m rows and n columns, where m (the number of equations) is greater than n (the number of variables). The best solution x (in the sense that it minimizes the error $|Ax - b|$) is the solution of the n equations $A^TAx = A^Tb$. If the columns of A are orthonormal vectors then $x = A^Tb$.

Eigenvalues and eigenvectors

The n eigenvalues λ_i and eigenvectors u_i of an $n \times n$ matrix A are the solutions of the equation $Au = \lambda u$. The eigenvalues are the zeros of the polynomial of degree n, $P_n(\lambda) = |A - \lambda I|$. If A is Hermitian then the eigenvalues λ_i are real and the eigenvectors u_i are mutually orthogonal. $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A.

$$\operatorname{Tr} A = \sum_i \lambda_i, \quad \text{also } |A| = \prod_i \lambda_i.$$

If S is a symmetric matrix, Λ is the diagonal matrix whose diagonal elements are the eigenvalues of S, and U is the matrix whose columns are the normalized eigenvectors of A, then

$$U^T S U = \Lambda \qquad \text{and} \qquad S = U \Lambda U^T.$$

If x is an approximation to an eigenvector of A then $x^T A x / (x^T x)$ (Rayleigh's quotient) is an approximation to the corresponding eigenvalue.

Commutators

$$[A,B] \equiv AB - BA$$

$$[A,B] = -[B,A]$$

$$[A,B]^\dagger = [B^\dagger, A^\dagger]$$

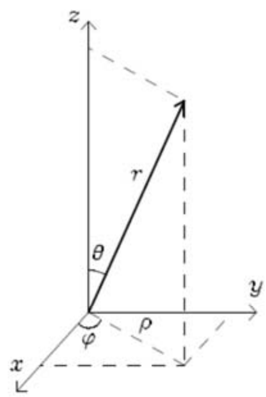
$$[A+B,C] = [A,C] + [B,C]$$

$$[AB,C] = A[B,C] + [A,C]B$$

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

Vector Calculus

Notation



ϕ is a scalar function of a set of position coordinates. In Cartesian coordinates $\phi = \phi(x, y, z)$; in cylindrical polar coordinates $\phi = \phi(\rho, \varphi, z)$; in spherical polar coordinates $\phi = \phi(r, \theta, \varphi)$; in cases with radial symmetry $\phi = \phi(r)$. A is a vector function whose components are scalar functions of the position coordinates: in Cartesian coordinates $A = iA_x + jA_y + kA_z$, where A_x, A_y, A_z are independent functions of x, y, z.

In Cartesian coordinates ∇ ('del') $\equiv i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z} \equiv \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix}$

$$\operatorname{grad}\phi = \nabla\phi, \qquad \operatorname{div} A = \nabla \cdot A, \qquad \operatorname{curl} A = \nabla \times A$$

Identities

$$\operatorname{grad}(\phi_1 + \phi_2) \equiv \operatorname{grad}\phi_1 + \operatorname{grad}\phi_2 \qquad \operatorname{div}(A_1 + A_2) \equiv \operatorname{div} A_1 + \operatorname{div} A_2$$

$$\operatorname{grad}(\phi_1\phi_2) \equiv \phi_1 \operatorname{grad}\phi_2 + \phi_2 \operatorname{grad}\phi_1$$

$$\operatorname{curl}(A_1 + A_2) \equiv \operatorname{curl} A_1 + \operatorname{curl} A_2$$

$$\operatorname{div}(\phi A) \equiv \phi \operatorname{div} A + (\operatorname{grad}\phi) \cdot A, \qquad \operatorname{curl}(\phi A) \equiv \phi \operatorname{curl} A + (\operatorname{grad}\phi) \times A$$

$$\operatorname{div}(A_1 \times A_2) \equiv A_2 \cdot \operatorname{curl} A_1 - A_1 \cdot \operatorname{curl} A_2$$

$$\operatorname{curl}(A_1 \times A_2) \equiv A_1 \operatorname{div} A_2 - A_2 \operatorname{div} A_1 + (A_2 \cdot \operatorname{grad})A_1 - (A_1 \cdot \operatorname{grad})A_2$$

$$\operatorname{div}(\operatorname{curl} A) \equiv 0, \qquad \operatorname{curl}(\operatorname{grad}\phi) \equiv 0$$

$$\operatorname{curl}(\operatorname{curl} A) \equiv \operatorname{grad}(\operatorname{div} A) - \operatorname{div}(\operatorname{grad} A) \equiv \operatorname{grad}(\operatorname{div} A) - \nabla^2 A$$

$$\operatorname{grad}(A_1 \cdot A_2) \equiv A_1 \times (\operatorname{curl} A_2) + (A_1 \cdot \operatorname{grad})A_2 + A_2 \times (\operatorname{curl} A_1) + (A_2 \cdot \operatorname{grad})A_1$$

Grad, Div, Curl and the Laplacian

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Cartesian Coordinates		$x=\rho\cos\varphi \quad y=\rho\sin\varphi \quad z=z$	$x=r\cos\varphi\sin\theta \quad y=r\sin\varphi\sin\theta$ $z=r\cos\theta$
Vector A	$A_x i + A_y j + A_z k$	$A_\rho\hat{\rho} + A_\varphi\hat{\varphi} + A_z\hat{z}$	$A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}$
Gradient $\nabla\phi$	$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$	$\frac{\partial\phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\hat{\varphi} + \frac{\partial\phi}{\partial z}\hat{z}$	$\frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\varphi}$
Divergence $\nabla\cdot A$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times A$	$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{\rho}\hat{\rho} & \hat{\varphi} & \frac{1}{\rho}\hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{r^2\sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian $\nabla^2\phi$	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

Complex Variables

Complex numbers

The complex number $z = x + iy = r(\cos\theta + i\sin\theta) = r\,e^{i(\theta+2n\pi)}$, where $i^2 = -1$ and n is an arbitrary integer. The real quantity r is the modulus of z and the angle θ is the argument of z. The complex conjugate of z is $z^* = x - iy = r(\cos\theta - i\sin\theta) = r\,e^{-i\theta}$; $\quad zz^* = |z|^2 = x^2 + y^2$

De Moivre's theorem

$$(\cos\theta + i\sin\theta)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

Power series for complex variables.

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \qquad \text{convergent for all finite } z$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad \text{convergent for all finite } z$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \qquad \text{convergent for all finite } z$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad \text{principal value of } \ln(1+z)$$

This last series converges both on and within the circle $|z| = 1$ except at the point $z = -1$.

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

This last series converges both on and within the circle $|z| = 1$ except at the points $z = \pm i$.

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + \cdots$$

This last series converges both on and within the circle $|z| = 1$ except at the point $z = -1$.

Trigonometric Formulae

$$\cos^2 A + \sin^2 A = 1 \qquad \sec^2 A - \tan^2 A = 1 \qquad \operatorname{cosec}^2 A - \cot^2 A = 1$$

$$\sin 2A = 2\sin A\cos A \qquad \cos 2A = \cos^2 A - \sin^2 A \qquad \tan 2A = \frac{2\tan A}{1-\tan^2 A}.$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B \qquad \cos A\cos B = \frac{\cos(A+B)+\cos(A-B)}{2}$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B \qquad \sin A\sin B = \frac{\cos(A-B)-\cos(A+B)}{2}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B} \qquad \sin A\cos B = \frac{\sin(A+B)+\sin(A-B)}{2}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \qquad \cos^2 A = \frac{1+\cos 2A}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \qquad \sin^2 A = \frac{1-\cos 2A}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \qquad \cos^3 A = \frac{3\cos A+\cos 3A}{4}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2} \qquad \sin^3 A = \frac{3\sin A-\sin 3A}{4}$$

Relations between sides and angles of any plane triangle

In a plane triangle with angles A, B, and C and sides opposite a, b, and c respectively,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter of circumscribed circle.}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a = b\cos C + c\cos B$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

$$\text{area} = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \sqrt{s(s-a)(s-b)(s-c)}, \qquad \text{where } s = \frac{1}{2}(a+b+c)$$

Hyperbolic Functions

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \text{valid for all } x$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{valid for all } x$$

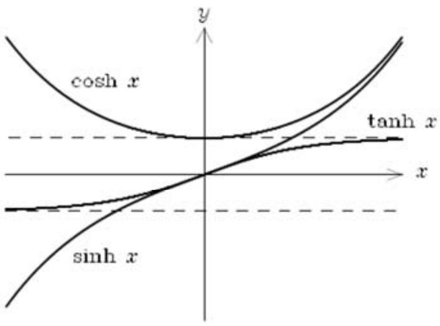
$\cosh ix = \cos x$ $\qquad$ $\cos ix = \cosh x$

$\sinh ix = i \sin x$ $\qquad$ $\sin ix = i \sinh x$

$\tanh x = \dfrac{\sinh x}{\cosh x}$ $\qquad$ $\operatorname{sech} x = \dfrac{1}{\cosh x}$

$\coth x = \dfrac{\cosh x}{\sinh x}$ $\qquad$ $\operatorname{cosech} x = \dfrac{1}{\sinh x}$

$\cosh^2 x - \sinh^2 x = 1$



For large positive x:

$\cosh x \approx \sinh x \to \dfrac{e^x}{2}$

$\tanh x \to 1$

For large negative x:

$\cosh x \approx -\sinh x \to \dfrac{e^{-x}}{2}$

$\tanh x \to -1$

Relations of the functions

$\sinh x = -\sinh(-x)$ $\qquad$ $\operatorname{sech} x = \operatorname{sech}(-x)$

$\cosh x = \cosh(-x)$ $\qquad$ $\operatorname{cosech} x = -\operatorname{cosech}(-x)$

$\tanh x = -\tanh(-x)$ $\qquad$ $\coth x = -\coth(-x)$

$$\sinh x = \frac{2\tanh(x/2)}{1 - \tanh^2(x/2)} = \frac{\tanh x}{\sqrt{1 - \tanh^2 x}} \qquad \cosh x = \frac{1 + \tanh^2(x/2)}{1 - \tanh^2(x/2)} = \frac{1}{\sqrt{1 - \tanh^2 x}}$$

$\tanh x = \sqrt{1 - \operatorname{sech}^2 x}$ $\qquad$ $\operatorname{sech} x = \sqrt{1 - \tanh^2 x}$

$\coth x = \sqrt{\operatorname{cosech}^2 x + 1}$ $\qquad$ $\operatorname{cosech} x = \sqrt{\coth^2 x - 1}$

$\sinh(x/2) = \sqrt{\dfrac{\cosh x - 1}{2}}$ $\qquad$ $\cosh(x/2) = \sqrt{\dfrac{\cosh x + 1}{2}}$

$$\tanh(x/2) = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$\sinh(2x) = 2 \sinh x \cosh x$ $\qquad$ $\tanh(2x) = \dfrac{2\tanh x}{1 + \tanh^2 x}$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

$\sinh(3x) = 3\sinh x + 4\sinh^3 x$ $\qquad$ $\cosh 3x = 4\cosh^3 x - 3\cosh x$

$$\tanh(3x) = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\sinh x + \sinh y = 2 \sinh \frac{1}{2}(x+y) \cosh \frac{1}{2}(x-y) \qquad \cosh x + \cosh y = 2 \cosh \frac{1}{2}(x+y) \cosh \frac{1}{2}(x-y)$$

$$\sinh x - \sinh y = 2 \cosh \frac{1}{2}(x+y) \sinh \frac{1}{2}(x-y) \qquad \cosh x - \cosh y = 2 \sinh \frac{1}{2}(x+y) \sinh \frac{1}{2}(x-y)$$

$$\sinh x \pm \cosh x = \frac{1 \pm \tanh(x/2)}{1 \mp \tanh(x/2)} = e^{\pm x}$$

$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}$$

$$\coth x \pm \coth y = \pm \frac{\sinh(x \pm y)}{\sinh x \sinh y}$$

Inverse functions

$$\sinh^{-1} \frac{x}{a} = \ln\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right) \qquad \text{for } -\infty < x < \infty$$

$$\cosh^{-1} \frac{x}{a} = \ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) \qquad \text{for } x \geq a$$

$$\tanh^{-1} \frac{x}{a} = \frac{1}{2} \ln\left(\frac{a+x}{a-x}\right) \qquad \text{for } x^2 < a^2$$

$$\coth^{-1} \frac{x}{a} = \frac{1}{2} \ln\left(\frac{x+a}{x-a}\right) \qquad \text{for } x^2 > a^2$$

$$\operatorname{sech}^{-1} \frac{x}{a} = \ln\left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2} - 1}\right) \qquad \text{for } 0 < x \leq a$$

$$\operatorname{cosech}^{-1} \frac{x}{a} = \ln\left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2} + 1}\right) \qquad \text{for } x \neq 0$$

Limits

$n^c x^n \to 0$ as $n \to \infty$ if $|x| < 1$ (any fixed c)

$x^n / n! \to 0$ as $n \to \infty$ (any fixed x)

$(1 + x/n)^n \to \mathrm{e}^x$ as $n \to \infty$, $x \ln x \to 0$ as $x \to 0$

If $f(a) = g(a) = 0$ then $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$ (l'Hôpital's rule)

Differentiation

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \cdots + {}^nC_r u^{(n-r)}v^{(r)} + \cdots + uv^{(n)}$$ Leibniz Theorem

where ${}^nC_r \equiv \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \cos x & \frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) &= \cosh x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) &= -\sin x & \frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) &= \sinh x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) &= \sec^2 x & \frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) &= \mathrm{sech}^2\, x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) &= \sec x \tan x & \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{sech}\, x) &= -\mathrm{sech}\, x \tanh x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) &= -\mathrm{cosec}^2\, x & \frac{\mathrm{d}}{\mathrm{d}x}(\coth x) &= -\mathrm{cosech}^2\, x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosec}\, x) &= -\mathrm{cosec}\, x \cot x & \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosech}\, x) &= -\mathrm{cosech}\, x \coth x
\end{aligned}$$

Integration

Standard forms

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad \text{for } n \neq -1$$

$$\int \frac{1}{x} \, dx = \ln x + c \qquad \int \ln x \, dx = x(\ln x - 1) + c$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c \qquad \int x e^{ax} \, dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right) + c$$

$$\int x \ln x \, dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2 - x^2} \, dx = \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) + c = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) + c \qquad \text{for } x^2 < a^2$$

$$\int \frac{1}{x^2 - a^2} \, dx = -\frac{1}{a} \coth^{-1}\left(\frac{x}{a}\right) + c = \frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right) + c \qquad \text{for } x^2 > a^2$$

$$\int \frac{x}{(x^2 \pm a^2)^n} \, dx = \frac{-1}{2(n-1)} \frac{1}{(x^2 \pm a^2)^{n-1}} + c \qquad \text{for } n \neq 1$$

$$\int \frac{x}{x^2 \pm a^2} \, dx = \frac{1}{2} \ln(x^2 \pm a^2) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} \, dx = \ln\left(x + \sqrt{x^2 \pm a^2}\right) + c$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} \, dx = \sqrt{x^2 \pm a^2} + c$$

$$\int \sqrt{a^2 - x^2} \, dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + c$$

$$\int_0^\infty \frac{1}{(1+x)x^p}\,\mathrm{d}x = \pi \operatorname{cosec} p\pi \qquad \text{for } p<1$$

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\int_{-\infty}^\infty \exp(-x^2/2\sigma^2)\,\mathrm{d}x = \sigma\sqrt{2\pi}$$

$$\int_{-\infty}^\infty x^n \exp(-x^2/2\sigma^2)\,\mathrm{d}x = \begin{cases} 1\times 3\times 5\times\cdots(n-1)\sigma^{n+1}\sqrt{2\pi} & \text{for } n\geq 2 \text{ and even} \\ 0 & \text{for } n\geq 1 \text{ and odd} \end{cases}$$

$$\begin{aligned}
\int \sin x\,\mathrm{d}x &= -\cos x + c & \int \sinh x\,\mathrm{d}x &= \cosh x + c \\
\int \cos x\,\mathrm{d}x &= \sin x + c & \int \cosh x\,\mathrm{d}x &= \sinh x + c \\
\int \tan x\,\mathrm{d}x &= -\ln(\cos x) + c & \int \tanh x\,\mathrm{d}x &= \ln(\cosh x) + c \\
\int \operatorname{cosec} x\,\mathrm{d}x &= \ln(\operatorname{cosec} x - \cot x) + c & \int \operatorname{cosech} x\,\mathrm{d}x &= \ln\left[\tanh(x/2)\right] + c \\
\int \sec x\,\mathrm{d}x &= \ln(\sec x + \tan x) + c & \int \operatorname{sech} x\,\mathrm{d}x &= 2\tan^{-1}(\mathrm{e}^x) + c \\
\int \cot x\,\mathrm{d}x &= \ln(\sin x) + c & \int \coth x\,\mathrm{d}x &= \ln(\sinh x) + c
\end{aligned}$$

$$\int \sin mx \sin nx\,\mathrm{d}x = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c \qquad \text{if } m^2\neq n^2$$

$$\int \cos mx \cos nx\,\mathrm{d}x = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + c \qquad \text{if } m^2\neq n^2$$

Standard substitutions

If the integrand is a function of:	substitute:
(a^2-x^2) or $\sqrt{a^2-x^2}$	$x=a\sin\theta$ or $x=a\cos\theta$
(x^2+a^2) or $\sqrt{x^2+a^2}$	$x=a\tan\theta$ or $x=a\sinh\theta$
(x^2-a^2) or $\sqrt{x^2-a^2}$	$x=a\sec\theta$ or $x=a\cosh\theta$

If the integrand is a rational function of $\sin x$ or $\cos x$ or both, substitute $t=\tan(x/2)$ and use the results:

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2}.$$

If the integrand is of the form:	substitute:
$\int \frac{\mathrm{d}x}{(ax+b)\sqrt{px+q}}$	$px+q=u^2$
$\int \frac{\mathrm{d}x}{(ax+b)\sqrt{px^2+qx+r}}$	$ax+b=\frac{1}{u}.$

Integration by parts

$$\int_a^b u \, \mathrm{d}v = uv\Big|_a^b - \int_a^b v \, \mathrm{d}u$$

Differentiation of an integral

If $f(x,\alpha)$ is a function of x containing a parameter α and the limits of integration a and b are functions of α then

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,\mathrm{d}x = f(b,\alpha)\frac{\mathrm{d}b}{\mathrm{d}\alpha} - f(a,\alpha)\frac{\mathrm{d}a}{\mathrm{d}\alpha} + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x,\alpha)\,\mathrm{d}x.$$

Special case,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(y)\,\mathrm{d}y = f(x).$$

Dirac δ-'function'

$$\delta(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[\mathrm{i}\omega(t-\tau)]\,\mathrm{d}\omega.$$

If $f(t)$ is an arbitrary function of t then $\int_{-\infty}^{\infty} \delta(t-\tau)f(t)\,\mathrm{d}t = f(\tau)$.

$\delta(t) = 0$ if $t \neq 0$, also $\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$

Reduction formulae

Factorials

$n! = n(n-1)(n-2)\ldots 1, \qquad 0! = 1.$

Stirling's formula for large n: $\ln(n!) \approx n\ln n - n$.

For any $p > -1$, $\int_0^{\infty} x^p \mathrm{e}^{-x}\,\mathrm{d}x = p\int_0^{\infty} x^{p-1}\mathrm{e}^{-x}\,\mathrm{d}x = p!$. $\qquad (-1/2)! = \sqrt{\pi}, \qquad (1/2)! = \sqrt{\pi}/2$, etc.

For any $p, q > -1$, $\int_0^1 x^p(1-x)^q\,\mathrm{d}x = \dfrac{p!q!}{(p+q+1)!}$.

Trigonometrical

If m, n are integers,

$$\int_0^{\pi/2} \sin^m\theta\,\cos^n\theta\,\mathrm{d}\theta = \frac{m-1}{m+n}\int_0^{\pi/2}\sin^{m-2}\theta\,\cos^n\theta\,\mathrm{d}\theta = \frac{n-1}{m+n}\int_0^{\pi/2}\sin^m\theta\,\cos^{n-2}\theta\,\mathrm{d}\theta$$

and can therefore be reduced eventually to one of the following integrals

$$\int_0^{\pi/2}\sin\theta\,\cos\theta\,\mathrm{d}\theta = \frac{1}{2}, \qquad \int_0^{\pi/2}\sin\theta\,\mathrm{d}\theta = 1, \qquad \int_0^{\pi/2}\cos\theta\,\mathrm{d}\theta = 1, \qquad \int_0^{\pi/2}\mathrm{d}\theta = \frac{\pi}{2}.$$

Other

If $I_n = \int_0^{\infty} x^n \exp(-\alpha x^2)\,\mathrm{d}x$ then $I_n = \dfrac{(n-1)}{2\alpha}I_{n-2}, \quad I_0 = \dfrac{1}{2}\sqrt{\dfrac{\pi}{\alpha}}, \quad I_1 = \dfrac{1}{2\alpha}$.

Diffusion (conduction) equation

$$\frac{\partial \psi}{\partial t} = \kappa \nabla^2 \psi$$

Wave equation

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

Bessel's equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - m^2) y = 0,$$

solutions of which are Bessel functions $J_m(x)$ of order m.

Series form of Bessel functions of the first kind

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{m+2k}}{k!(m+k)!} \qquad \text{(integer } m\text{)}.$$

The same general form holds for non-integer $m > 0$.

Laplace's equation

$$\nabla^2 u = 0$$

If expressed in two-dimensional polar coordinates (see section 4), a solution is

$$u(\rho, \varphi) = \left[A\rho^n + B\rho^{-n}\right]\left[C\exp(in\varphi) + D\exp(-in\varphi)\right]$$

where A, B, C, D are constants and n is a real integer.

If expressed in three-dimensional polar coordinates (see section 4) a solution is

$$u(r, \theta, \varphi) = \left[Ar^l + Br^{-(l+1)}\right]P_l^m\left[C\sin m\varphi + D\cos m\varphi\right]$$

where l and m are integers with $l \geq |m| \geq 0$; A, B, C, D are constants;

$$P_l^m(\cos\theta) = \sin^{|m|}\theta\left[\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}\right]^{|m|}P_l(\cos\theta)$$

is the associated Legendre polynomial.

$$P_l^0(1) = 1.$$

If expressed in cylindrical polar coordinates (see section 4), a solution is

$$u(\rho, \varphi, z) = J_m(n\rho)\left[A\cos m\varphi + B\sin m\varphi\right]\left[C\exp(nz) + D\exp(-nz)\right]$$

where m and n are integers; A, B, C, D are constants.

Functions of Several Variables

If $\phi = f(x, y, z, \ldots)$ then $\dfrac{\partial \phi}{\partial x}$ implies differentiation with respect to x keeping $y, z, \ldots$ constant.

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{d}z + \cdots \quad \text{and} \quad \delta\phi \approx \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z + \cdots$$

where $x, y, z, \ldots$ are independent variables. $\dfrac{\partial \phi}{\partial x}$ is also written as $\left(\dfrac{\partial \phi}{\partial x}\right)_{y,\ldots}$ or $\left.\dfrac{\partial \phi}{\partial x}\right|_{y,\ldots}$ when the variables kept constant need to be stated explicitly.

If ϕ is a well-behaved function then $\dfrac{\partial^2 \phi}{\partial x\,\partial y} = \dfrac{\partial^2 \phi}{\partial y\,\partial x}$ etc.

If $\phi = f(x, y)$,

$$\left(\frac{\partial \phi}{\partial x}\right)_y = \frac{1}{\left(\dfrac{\partial x}{\partial \phi}\right)_y}, \qquad \left(\frac{\partial \phi}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_\phi \left(\frac{\partial y}{\partial \phi}\right)_x = -1.$$

Taylor series for two variables

If $\phi(x, y)$ is well-behaved in the vicinity of $x = a, y = b$ then it has a Taylor series

$$\phi(x, y) = \phi(a + u, b + v) = \phi(a, b) + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + \frac{1}{2!}\left(u^2\frac{\partial^2 \phi}{\partial x^2} + 2uv\frac{\partial^2 \phi}{\partial x\,\partial y} + v^2\frac{\partial^2 \phi}{\partial y^2}\right) + \cdots$$

where $x = a + u, y = b + v$ and the differential coefficients are evaluated at $x = a, \quad y = b$

Stationary points

A function $\phi = f(x, y)$ has a stationary point when $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \phi}{\partial y} = 0$. Unless $\dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{\partial^2 \phi}{\partial x\,\partial y} = 0$, the following conditions determine whether it is a minimum, a maximum or a saddle point.

$$\left.\begin{array}{ll} \text{Minimum:} & \dfrac{\partial^2 \phi}{\partial x^2} > 0, \text{ or } \dfrac{\partial^2 \phi}{\partial y^2} > 0, \\ \text{Maximum:} & \dfrac{\partial^2 \phi}{\partial x^2} < 0, \text{ or } \dfrac{\partial^2 \phi}{\partial y^2} < 0, \end{array}\right\} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} > \left(\frac{\partial^2 \phi}{\partial x\,\partial y}\right)^2$$

$$\text{Saddle point:} \quad \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} < \left(\frac{\partial^2 \phi}{\partial x\,\partial y}\right)^2$$

If $\dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{\partial^2 \phi}{\partial x\,\partial y} = 0$ the character of the turning point is determined by the next higher derivative.

Changing variables: the chain rule

If $\phi = f(x, y, \ldots)$ and the variables $x, y, \ldots$ are functions of independent variables $u, v, \ldots$ then

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + \cdots$$
$$\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial v} + \cdots$$

etc.

Changing variables in surface and volume integrals – Jacobians

If an area A in the x, y plane maps into an area A' in the u, v plane then

$$\int_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{A'} f(u,v) J\,\mathrm{d}u\,\mathrm{d}v \quad \text{where} \quad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian J is also written as $\dfrac{\partial(x,y)}{\partial(u,v)}$. The corresponding formula for volume integrals is

$$\int_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{V'} f(u,v,w) J\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \qquad \text{where now} \qquad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Fourier Series and Transforms

Fourier series

If $y(x)$ is a function defined in the range $-\pi \leq x \leq \pi$ then

$$y(x) \approx c_0 + \sum_{m=1}^{M} c_m \cos mx + \sum_{m=1}^{M'} s_m \sin mx$$

where the coefficients are

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x)\,\mathrm{d}x$$

$$c_m = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos mx\,\mathrm{d}x \qquad (m = 1, \ldots, M)$$

$$s_m = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin mx\,\mathrm{d}x \qquad (m = 1, \ldots, M')$$

with convergence to $y(x)$ as $M, M' \to \infty$ for all points where $y(x)$ is continuous.

Fourier series for other ranges

Variable t, range $0 \leq t \leq T$, (i.e., a periodic function of time with period T, frequency $\omega = 2\pi/T$).

$$y(t) \approx c_0 + \sum c_m \cos m\omega t + \sum s_m \sin m\omega t$$

where

$$c_0 = \frac{\omega}{2\pi} \int_0^T y(t)\,\mathrm{d}t, \quad c_m = \frac{\omega}{\pi} \int_0^T y(t) \cos m\omega t\,\mathrm{d}t, \quad s_m = \frac{\omega}{\pi} \int_0^T y(t) \sin m\omega t\,\mathrm{d}t.$$

Variable x, range $0 \leq x \leq L$,

$$y(x) \approx c_0 + \sum c_m \cos \frac{2m\pi x}{L} + \sum s_m \sin \frac{2m\pi x}{L}$$

where

$$c_0 = \frac{1}{L} \int_0^L y(x)\,\mathrm{d}x, \quad c_m = \frac{2}{L} \int_0^L y(x) \cos \frac{2m\pi x}{L}\,\mathrm{d}x, \quad s_m = \frac{2}{L} \int_0^L y(x) \sin \frac{2m\pi x}{L}\,\mathrm{d}x.$$

Fourier series for odd and even functions

If $y(x)$ is an *odd* (anti-symmetric) function [i.e., $y(-x) = -y(x)$] defined in the range $-\pi \le x \le \pi$, then only sines are required in the Fourier series and $s_m = \frac{2}{\pi}\int_0^\pi y(x)\sin mx\,\mathrm{d}x$. If, in addition, $y(x)$ is symmetric about $x = \pi/2$, then the coefficients s_m are given by $s_m = 0$ (for m even), $s_m = \frac{4}{\pi}\int_0^{\pi/2} y(x)\sin mx\,\mathrm{d}x$ (for m odd). If $y(x)$ is an *even* (symmetric) function [i.e., $y(-x) = y(x)$] defined in the range $-\pi \le x \le \pi$, then only constant and cosine terms are required in the Fourier series and $c_0 = \frac{1}{\pi}\int_0^\pi y(x)\,\mathrm{d}x$, $c_m = \frac{2}{\pi}\int_0^\pi y(x)\cos mx\,\mathrm{d}x$. If, in addition, $y(x)$ is anti-symmetric about $x = \frac{\pi}{2}$, then $c_0 = 0$ and the coefficients c_m are given by $c_m = 0$ (for m even), $c_m = \frac{4}{\pi}\int_0^{\pi/2} y(x)\cos mx\,\mathrm{d}x$ (for m odd).

[These results also apply to Fourier series with more general ranges provided appropriate changes are made to the limits of integration.]

Complex form of Fourier series

If $y(x)$ is a function defined in the range $-\pi \le x \le \pi$ then

$$y(x) \approx \sum_{-M}^{M} C_m\,\mathrm{e}^{\mathrm{i}mx}, \quad C_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(x)\,\mathrm{e}^{-\mathrm{i}mx}\,\mathrm{d}x$$

with m taking all integer values in the range $\pm M$. This approximation converges to $y(x)$ as $M \to \infty$ under the same conditions as the real form.

For other ranges the formulae are:

Variable t, range $0 \le t \le T$, frequency $\omega = 2\pi/T$,

$$y(t) = \sum_{-\infty}^{\infty} C_m\,\mathrm{e}^{\mathrm{i}m\omega t}, \quad C_m = \frac{\omega}{2\pi}\int_0^T y(t)\,\mathrm{e}^{-\mathrm{i}m\omega t}\,\mathrm{d}t.$$

Variable x', range $0 \le x' \le L$,

$$y(x') = \sum_{-\infty}^{\infty} C_m\,\mathrm{e}^{\mathrm{i}2m\pi x'/L}, \quad C_m = \frac{1}{L}\int_0^L y(x')\,\mathrm{e}^{-\mathrm{i}2m\pi x'/L}\,\mathrm{d}x'.$$

Discrete Fourier series

If $y(x)$ is a function defined in the range $-\pi \le x \le \pi$ which is sampled in the $2N$ equally spaced points $x_n = nx/N$ $[n = -(N-1)\ldots N]$, then

$$\begin{aligned} y(x_n) = c_0 &+ c_1\cos x_n + c_2\cos 2x_n + \cdots + c_{N-1}\cos(N-1)x_n + c_N\cos Nx_n \\ &+ s_1\sin x_n + s_2\sin 2x_n + \cdots + s_{N-1}\sin(N-1)x_n + s_N\sin Nx_n \end{aligned}$$

where the coefficients are

$$\begin{aligned} c_0 &= \frac{1}{2N}\sum y(x_n) \\ c_m &= \frac{1}{N}\sum y(x_n)\cos mx_n && (m = 1,\ldots,N-1) \\ c_N &= \frac{1}{2N}\sum y(x_n)\cos Nx_n \\ s_m &= \frac{1}{N}\sum y(x_n)\sin mx_n && (m = 1,\ldots,N-1) \\ s_N &= \frac{1}{2N}\sum y(x_n)\sin Nx_n \end{aligned}$$

each summation being over the $2N$ sampling points x_n.

Fourier transforms

If $y(x)$ is a function defined in the range $-\infty \le x \le \infty$ then the Fourier transform $\widehat{y}(\omega)$ is defined by the equations

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{y}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega, \qquad \widehat{y}(\omega) = \int_{-\infty}^{\infty} y(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.$$

If ω is replaced by $2\pi f$, where f is the frequency, this relationship becomes

$$y(t) = \int_{-\infty}^{\infty} \widehat{y}(f)\,\mathrm{e}^{\mathrm{i}2\pi f t}\,\mathrm{d}f, \qquad \widehat{y}(f) = \int_{-\infty}^{\infty} y(t)\,\mathrm{e}^{-\mathrm{i}2\pi f t}\,\mathrm{d}t.$$

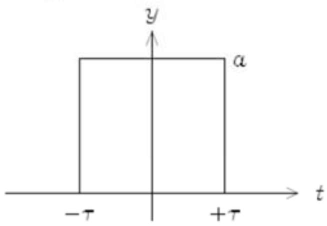
If $y(t)$ is symmetric about $t = 0$ then

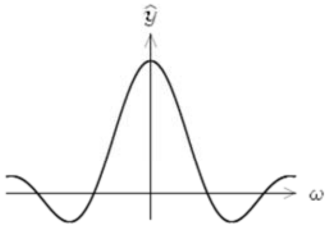
$$y(t) = \frac{1}{\pi}\int_{0}^{\infty} \widehat{y}(\omega)\cos\omega t\,\mathrm{d}\omega, \qquad \widehat{y}(\omega) = 2\int_{0}^{\infty} y(t)\cos\omega t\,\mathrm{d}t.$$

If $y(t)$ is anti-symmetric about $t = 0$ then

$$y(t) = \frac{1}{\pi}\int_{0}^{\infty} \widehat{y}(\omega)\sin\omega t\,\mathrm{d}\omega, \qquad \widehat{y}(\omega) = 2\int_{0}^{\infty} y(t)\sin\omega t\,\mathrm{d}t.$$

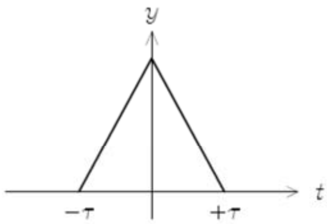
Specific cases

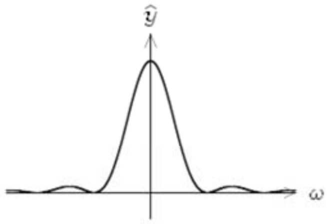




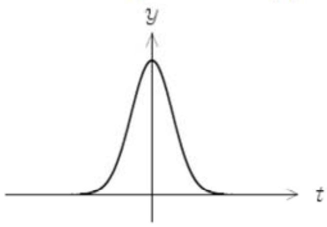
$$\left.\begin{aligned} y(t) &= a, & |t| &\le \tau \\ &= 0, & |t| &> \tau \end{aligned}\right\} \quad \textit{('Top Hat')}, \qquad \widehat{y}(\omega) = 2a\frac{\sin\omega\tau}{\omega} \equiv 2a\tau\,\mathrm{sinc}(\omega\tau)$$

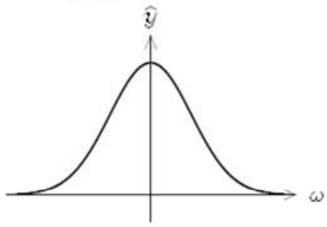
where $\quad \mathrm{sinc}(x) = \dfrac{\sin(x)}{x}$





$$\left.\begin{aligned} y(t) &= a(1 - |t|/\tau), & |t| &\le \tau \\ &= 0, & |t| &> \tau \end{aligned}\right\} \quad \textit{('Saw-tooth')}, \qquad \widehat{y}(\omega) = \frac{2a}{\omega^2\tau}(1 - \cos\omega\tau) = a\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2}\right)$$





$$y(t) = \exp(-t^2/t_0^2) \quad \textit{(Gaussian)}, \qquad \widehat{y}(\omega) = t_0\sqrt{\pi}\exp\left(-\omega^2 t_0^2/4\right)$$

$$y(t) = f(t)\,\mathrm{e}^{\mathrm{i}\omega_0 t} \quad \textit{(modulated function)}, \qquad \widehat{y}(\omega) = \widehat{f}(\omega - \omega_0)$$

$$y(t) = \sum_{m=-\infty}^{\infty} \delta(t - m\tau) \quad \textit{(sampling function)} \qquad \widehat{y}(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n/\tau)$$

Convolution theorem

If $z(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)\,\mathrm{d}\tau \quad \equiv x(t) * y(t)$ then $\quad \widehat{z}(\omega) = \widehat{x}(\omega)\,\widehat{y}(\omega)$.

Conversely, $\widehat{xy} = \widehat{x} * \widehat{y}$.

Laplace Transforms

If $y(t)$ is a function defined for $t \geq 0$, the Laplace transform $\overline{y}(s)$ is defined by the equation

$$\overline{y}(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt$$

Function $y(t)$ $(t > 0)$	Transform $\overline{y}(s)$	
$\delta(t)$	1	Delta function
$\theta(t)$	$\frac{1}{s}$	Unit step function
t^n	$\frac{n!}{s^{n+1}}$	
$t^{1/2}$	$\frac{1}{2}\sqrt{\frac{\pi}{s^3}}$	
$t^{-1/2}$	$\sqrt{\frac{\pi}{s}}$	
e^{-at}	$\frac{1}{(s+a)}$	
$\sin \omega t$	$\frac{\omega}{(s^2+\omega^2}$	
$\cos \omega t$	$\frac{s}{(s^2+\omega^2)}$	
$\sinh \omega t$	$\frac{\omega}{(s^2-\omega^2)}$	
$\cosh \omega t$	$\frac{s}{(s^2-\omega^2)}$	
$e^{-at}y(t)$	$\overline{y}(s+a)$	
$y(t-\tau)\,\theta(t-\tau)$	$e^{-s\tau}\,\overline{y}(s)$	
$ty(t)$	$-\frac{d\overline{y}}{ds}$	
$\frac{dy}{dt}$	$s\overline{y}(s) - y(0)$	
$\frac{d^n y}{dt^n}$	$s^n\overline{y}(s) - s^{n-1}y(0) - s^{n-2}\left[\frac{dy}{dt}\right]_0 \cdots - \left[\frac{d^{n-1}y}{dt^{n-1}}\right]_0$	
$\int_0^t y(\tau)\, d\tau$	$\frac{\overline{y}(s)}{s}$	
$\left.\begin{matrix}\int_0^t x(\tau)\, y(t-\tau)\, d\tau \\ \int_0^t x(t-\tau)\, y(\tau)\, d\tau\end{matrix}\right\}$	$\overline{x}(s)\,\overline{y}(s)$	Convolution theorem

[Note that if $y(t) = 0$ for $t < 0$ then the Fourier transform of $y(t)$ is $\hat{y}(\omega) = \overline{y}(i\omega)$.]

Numerical Analysis

Finding the zeros of equations

If the equation is $y = f(x)$ and x_n is an approximation to the root then either

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \text{(Newton)}$$

$$\text{or, } x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \qquad \text{(Linear interpolation)}$$

are, in general, better approximations.

Numerical integration of differential equations

If $\frac{dy}{dx} = f(x, y)$ then

$$y_{n+1} = y_n + hf(x_n, y_n) \quad \text{where } h = x_{n+1} - x_n \qquad \text{(Euler method)}$$

Putting $y^*_{n+1} = y_n + hf(x_n, y_n)$ (improved Euler method)

then $y_{n+1} = y_n + \dfrac{h[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]}{2}$

Numerical evaluation of definite integrals

Trapezoidal rule

The interval of integration is divided into n equal sub-intervals, each of width h; then

$$\int_a^b f(x)\, dx \approx h\left[c\frac{1}{2}f(a) + f(x_1) + \cdots + f(x_j) + \cdots + \frac{1}{2}f(b)\right]$$

where $h = (b - a)/n$ and $x_j = a + jh$.

Simpson's rule

The interval of integration is divided into an even number (say $2n$) of equal sub-intervals, each of width $h = (b - a)/2n$; then

$$\int_a^b f(x)\, dx \approx \frac{h}{3}[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(b)]$$

Gauss's integration formulae

These have the general form $\int_{-1}^{1} y(x)\, dx \approx \sum_1^n c_i y(x_i)$

For $n = 2$: $x_i = \pm 0{\cdot}5773$; $c_i = 1, 1$ (exact for any cubic).

For $n = 3$: $x_i = -0{\cdot}7746, 0{\cdot}0, 0{\cdot}7746$; $c_i = 0{\cdot}555, 0{\cdot}888, 0{\cdot}555$ (exact for any quintic).